

On the linearity of lattices in affine buildings and ergodicity of the singular Cartan flow

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Abstract

Let X be a locally finite irreducible affine building of dimension ≥ 2 and $\Gamma \leq \text{Aut}(X)$ be a discrete group acting cocompactly. The goal of this paper is to address the following question: When is Γ linear? More generally, when does Γ admit a finite-dimensional representation with infinite image over a commutative unital ring? If X is the Bruhat–Tits building of a simple algebraic group over a local field and if Γ is an arithmetic lattice, then Γ is clearly linear. We prove that if X is of type \tilde{A}_2 , then the converse holds. In particular, cocompact lattices in exotic \tilde{A}_2 -buildings are non-linear. As an application, we obtain the first infinite family of lattices in exotic \tilde{A}_2 -buildings of arbitrarily large thickness, providing also a partial answer to a question of W. Kantor from 1986. We also show that if X is Bruhat–Tits of arbitrary type, then the linearity of Γ implies that Γ is virtually contained in the linear part of the automorphism group of X ; in particular Γ is an arithmetic lattice. The proofs are based on the machinery of algebraic representations of ergodic systems recently developed by U. Bader and A. Furman. The implementation of that tool in the present context requires the geometric construction of a suitable ergodic Γ -space attached to the building X , which we call the *singular Cartan flow*.

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1 Introduction

1.1 Are lattices in affine buildings linear?

Let X be a locally finite affine building with a cocompact automorphism group, and $\Gamma < \text{Aut}(X)$ be a lattice. The goal of this work is to address the following question: When does Γ admit a faithful finite-dimensional linear representations over a field, or more generally over a commutative ring with unity? A group admitting such a representation is called **linear**.

Let us review some known facts concerning that question.

The most important locally finite affine buildings are the so-called **Bruhat–Tits buildings**. Such a building X is associated with a semi-simple algebraic group \mathbf{G} over a local field k . The dimension of X coincides with the k -rank of \mathbf{G} . When \mathbf{G} is k -simple, the building X is irreducible. In that case, if we assume moreover that $\dim(X) \geq 2$, celebrated theorems by Margulis and Venkatarmana ensure that lattices in $\mathbf{G}(k)$ are all arithmetic. It is tempting to deduce that lattices in Bruhat–Tits buildings of dimension ≥ 2 are classified, but this is actually not the case. Indeed, the full automorphism group of the Bruhat–Tits building X of $\mathbf{G}(k)$ is isomorphic to $\text{Aut}(\mathbf{G}(k))$ (assuming that the k -rank of \mathbf{G} is ≥ 2), which is an extension of the group $\text{Aut}(\mathbf{G})(k)$ of algebraic automorphisms over k by a subgroup $\text{Aut}(k)$ (see [12] and Proposition C.1 below). As long as $\text{Aut}(k)$ is infinite — which happens if and only if the local field k is of positive characteristic — a lattice in $\text{Aut}(X)$ can potentially have infinite image in $\text{Aut}(k)$ and, hence, fail to satisfy the hypotheses of the arithmeticity theorem. A lattice in $\text{Aut}(X)$ with infinite image in $\text{Aut}(k)$ will be called a **Galois lattice**. Galois lattices exist in rank one (see Proposition C.4 below), while their existence in higher rank is an open question at the time of writing.

By the seminal work of Tits (see the last corollary in [67]), irreducible locally finite affine building of dimension ≥ 3 are all Bruhat–Tits buildings of simple algebraic groups over local fields. This is not the case in low dimension.

If $\dim(X) = 1$, namely if X is a tree with vertex degrees ≥ 2 , any uniform lattice in $\text{Aut}(X)$ is linear, indeed virtually free. However, a non-uniform lattice can be non-linear.

If $\dim(X) = 2$ and X is reducible, then X is the product of two trees. In that case Burger and Mozes [16] constructed celebrated examples of uniform lattices $\Gamma < \text{Aut}(X)$ that are simple groups, and can thus not be linear since they fail to be residually finite. It is still unknown whether an irreducible lattice in a product of more than two topologically simple groups, each acting cocompactly on a tree, can be non-linear.

If $\dim(X) = 2$ and X is irreducible, but not Bruhat–Tits, then we say that X is **exotic**. Some of the known constructions of exotic affine buildings and lattices acting on them are referenced in §1.3 below.

The simplest case of exotic affine buildings, which has been the most studied so far and will be of core interest in this paper, is that of buildings of type \tilde{A}_2 . Let us recall that (locally finite) \tilde{A}_2 -buildings may be characterized among simply connected 2-dimensional simplicial complexes by the property that the link of every vertex is the incidence graph of a (finite) projective plane; this follows from [28, Th. 7.3]. From the classification, it follows that the locally finite Bruhat–Tits buildings of type \tilde{A}_2 are precisely those associated with $\text{PGL}_3(D)$, where D is a finite dimensional division algebra over a local field k . Thus a locally finite \tilde{A}_2 -building is exotic if and only if it is not isomorphic to the Bruhat–Tits building of such a group $\text{PGL}_3(D)$.

Our first main result, whose proof occupies the major part of this paper, is the following.

Theorem 1.1. *Let X be a locally finite \tilde{A}_2 -building and $\Gamma < \text{Aut}(X)$ be a discrete group acting cocompactly. Assume that X is not isomorphic to the building associated to $\text{PGL}_3(D)$, with D a finite dimensional division algebra over a local field.*

Then, for any commutative ring R with unity and any $n \geq 1$, any homomorphism $\Gamma \rightarrow \text{GL}_n(R)$ has a finite image.

In Appendix C, we shall also establish a similar result for Galois lattices in arbitrary Bruhat–Tits buildings of dimension ≥ 2 .

Theorem 1.2. *Let \mathbf{G} be a k -simple algebraic group defined over a local field k , of k -rank ≥ 2 . Let Γ be a lattice in $\text{Aut}(\mathbf{G}(k))$.*

If Γ has infinite image in $\text{Out}(\mathbf{G}(k))$, then for any commutative unital ring R and any $n \geq 1$, any homomorphism $\Gamma \rightarrow \text{GL}_n(R)$ has a finite image.

Combining these two results with the aforementioned Arithmeticity Theorem, we draw the following consequence.

Corollary 1.3. *Let X be a locally finite \tilde{A}_2 -building and $\Gamma < \text{Aut}(X)$ be a discrete group acting cocompactly.*

Then Γ admits a representation with infinite image of dimension $n \geq 1$ over a commutative unital ring if and only if X is Bruhat–Tits and Γ is arithmetic.

We mention that by an unpublished work of Yehuda Shalom and Tim Steger we know that every proper normal subgroup of a lattice in an \tilde{A}_2 -building is of finite index. Clearly if the lattice is linear then finite index normal subgroups are in abundance. The question of finding finite quotients of lattices in possibly exotic affine buildings was already asked by W. Kantor [43, Problem C.6.2]. Some computer experiments we held indicate that some non-linear lattices might have no normal subgroups at all. We suggest the following.

Conjecture 1.4. *Let X be an affine building of type \tilde{A}_2 . Assume that X is not isomorphic to the building associated to $\text{PGL}_3(D)$, where D is a finite dimensional division algebra over a local field. Let Γ be a lattice in $\text{Aut}(X)$. Then Γ is virtually simple.*

It should be emphasized that, while there are several known constructions of locally finite \tilde{A}_2 -buildings with lattices, it is generally a delicate problem to determine whether or not a given such building arising from one of those constructions is Bruhat–Tits. Actually, until now, the only examples that could be proved to be exotic are buildings of thickness ≤ 10 , and in most cases the proof of exoticity relied on computer calculations. As an application of Theorem 1.1, presented in Appendix D, we will establish an exoticity criterion for a large concrete family of \tilde{A}_2 -lattices which was first defined by Ronan [56, §3] and Kantor [43, §C.3] and more recently investigated by Essert [33]. In particular, we obtain the first infinite family of \tilde{A}_2 -lattices in exotic buildings of arbitrarily large thickness (see Corollary D.10), and provide a partial answer to a question of W. Kantor, who wrote in the remarks following Corollary C.3.2 in [43]: “It would be very interesting to know which, if any, of these buildings are “classical” ones obtained from $\text{PSL}_3(K)$ for complete local (skew) fields K .”

1.2 Proof ingredients

The proofs of both Theorems 1.1 and 1.2 establish the contrapositive assertion, and thus start by assuming that the lattice Γ has a linear representation with infinite image over a ring.

The first step of the proof is a reduction showing that the given representation can be assumed to range in a k -simple algebraic group \mathbf{G} over a local field k , with an unbounded and Zariski dense image. That step is actually a consequence of an abstract statement of independent interest, valid at a high level of generality, and presented in Appendix A.

At that stage, the proof of Theorem 1.1 reduces to that of the following.

Theorem 1.5. *Let X be a locally finite \tilde{A}_2 -building and $\Gamma < \text{Aut}(X)$ a discrete group acting cocompactly. Assume that there exist a local field k , a connected adjoint k -simple k -algebraic group \mathbf{G} and a group homomorphism $\Gamma \rightarrow \mathbf{G}(k)$ with an unbounded and Zariski dense image. Then X is a Bruhat–Tits building.*

Our strategy to approach Theorem 1.5 is inspired by the recent proof of the Margulis Superrigidity Theorem by the first author and A. Furman, developed in [5]. The tools developed in that work are designed to study a representation $\rho: \Gamma \rightarrow \mathbf{J}(k)$ of an arbitrary countable group Γ to k -simple algebraic group over a local field k , with Zariski dense image. A key notion from [5] is that of a **gate** (see §8.1 below for more details). This gate is an algebraic variety canonically associated with any standard Borel space Y equipped with a measure-class preserving ergodic action of Γ . The gate moreover provides a natural representation from $\text{Aut}_\Gamma(Y)$ to a subquotient of the algebraic group $\mathbf{J}(k)$, which is non-trivial under suitable assumptions on Γ and Y .

Therefore, in order to prove Theorem 1.5, we need to provide an ergodic Γ -space Y such that $\text{Aut}_\Gamma(Y)$ does not admit any non-trivial representation. The space Y we shall construct will be called the **singular Cartan flow**. In order to give a better understanding of its construction, let us first explain what this space is in the ‘classical’ case when Γ is a lattice in $\text{SL}_3(k)$, where k is a local field.

Let A be the subgroup of $G = \text{SL}_3(k)$ consisting of diagonal matrices, and S the singular torus

$$S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} \mid a \in k \right\}$$

Then we consider the Γ -spaces G/A and G/S (or the actions of A and S on $\Gamma \backslash G$). Ergodicity is then provided by the Howe–Moore Theorem. The gate of the space $Y = G/S$ gives a representation of $N_G(S)/S \simeq \text{PGL}_2(k)$.

Now let us go back to the setting of Theorem 1.5, where Γ is a lattice of an \tilde{A}_2 -building X that can a priori be exotic. In particular the whole group $\text{Aut}(X)$ could be discrete, and the Γ -spaces we need to construct can thus hardly be homogeneous spaces of a non-discrete locally compact envelope of Γ . The spaces we shall construct are of geometric nature; the measures will be constructed explicitly by hand, and a crucial step in our proof will be to establish their ergodicity.

In order to define a geometric analogue of the space G/A in the classical case mentioned above, we consider the set \mathcal{F} consisting of all simplicial embeddings of the Coxeter complex Σ of type \tilde{A}_2 to X . The group \mathbf{Z}^2 acts on the set \mathcal{F} via its action on Σ , while Γ acts on \mathcal{F} via its action on X . In particular these two actions commute, and we get a \mathbf{Z}^2 -action on the compact space \mathcal{F}/Γ . The latter action is called the **Cartan Flow**. A measure on \mathcal{F} is constructed in §6.2.

A geometric analogue of the space G/S from the classical picture is more technical to realize. It is a kind of geodesic flow in the singular direction : the group \mathbf{Z} acts by translating

singular geodesic. The space we consider however is larger than this: we consider all the geodesics which are parallel to a given one, and translate them at once. This gives an action of \mathbf{Z} on a space \mathcal{S} , which we call the **singular Cartan flow**. We again define a natural measure on this space \mathcal{S} (see §6.3 for more details).

A fundamental step in our proof of Theorem 1.5 is the following (see Theorem 7.1 and Theorem 7.3 for more precise statements).

Theorem 1.6. *The Cartan flow and the singular Cartan flow are ergodic.*

As mentioned above, those flows are a priori not homogeneous spaces of some locally compact groups; in particular the ergodicity cannot be deduced from the Howe–Moore Theorem as it is done in the classical picture. The relevant tool happens to be one originally used in the study of the geodesic flow on manifolds with variable negative curvature, namely Hopf’s argument [39] (see also [30] for a nice introduction). In Appendix B we give an abstract version of the Hopf argument which is suited for our purpose. The Cartan flow is not needed for the end of the proof, but it is an intermediate step in the proof of the ergodicity of the singular Cartan flow.

If Y denotes the singular Cartan flow, the gate theory provides us with a linear representation of $\text{Aut}_\Gamma(Y)$ (the latter group is $\text{PGL}_2(k)$ in the classical case). It appears that the group $\text{Aut}_\Gamma(Y)$ naturally contains a group that has been studied in the axiomatic theory of projective geometry: it is the so-called **projectivity group** of the projective plane at infinity of X . This projectivity group has a lot more structure than Γ : it acts on a locally finite tree T (a so-called *panel-tree* of X) with a 3-transitive action on its boundary. The final step in the proof of Theorem 1.5 will be deduced from the following result of independent interest (see Theorem 4.7):

Theorem 1.7. *Let X be a locally finite \tilde{A}_2 -building and P be the closure of the projectivity group in the locally compact group $\text{Aut}(T)$. If an open subgroup of P admits a faithful continuous representation $P \rightarrow \text{GL}_n(k)$, where k is a local field, then X is a Bruhat–Tits building.*

It should be underlined that, while all the other steps of the proof generalize with only small modifications to Euclidean buildings of other types than \tilde{A}_2 , we do not know how to extend Theorem 1.7 to \tilde{B}_2 or \tilde{G}_2 buildings. Indeed the proof of the latter relies on a result on projective planes due to A. Schleiermacher (Theorem 3.2), whose analogue for generalized quadrangles and generalized hexagons is still unknown.

Let us now briefly discuss the proof of Theorem 1.2, which is presented in Appendix C. It also relies on the Bader–Furman machinery from [5]. The starting point of that proof is the observation that the argument for Theorem 1.1 sketched above also provides relevant information when X is the Bruhat–Tits building of $\text{PGL}_3(D)$. Indeed, while in that case the projectivity group is isomorphic to $\text{PSL}_2(D)$, and is thus linear by hypothesis, it turns out that the group $\text{Aut}_\Gamma(Y)$ of automorphisms of the Cartan flow commuting with Γ considered above can be strictly larger than the projectivity group. More precisely, if Γ is a Galois lattice, then Γ has an infinite (hence non-virtually abelian by property (T)) image in the quotient $\text{Aut}(X)/\text{PGL}_3(D)$, which is virtually the group of field automorphisms of X . It then turns out that $\text{Aut}_\Gamma(Y)/\text{PSL}_2(D)$ is also a non-abelian group of field automorphisms of $\text{PSL}_2(D)$. On the other hand, the gate theory provides us with a continuous faithful representation of the locally compact group $\text{Aut}_\Gamma(Y)$, and by the general structure theory

of linear locally compact groups from [24], this implies that $\text{Aut}_\Gamma(Y)/\text{PSL}_2(D)$ must in fact be virtually abelian. This shows that if Γ is linear, then it cannot be a Galois lattice. This part of the argument does not require X to be of type \tilde{A}_2 , and is actually valid for arbitrary irreducible Bruhat–Tits buildings of dimension ≥ 2 . Indeed the relevant Γ -spaces are indeed homogeneous spaces of the non-discrete group $\text{Aut}(X)$, and their ergodicity can be established by a suitable application of the Howe–Moore theorem.

1.3 A review of exotic 2-dimensional buildings with lattices

The aforementioned classification of irreducible affine buildings of dimension ≥ 3 was completed in the early days of the 1980’s. The existence of exotic 2-dimensional affine buildings came soon after. Some of them have no non-trivial automorphism whatsoever (see e.g. [57]); this is in particular the case of *generic* \tilde{A}_2 -buildings (in a suitably defined Baire categorical sense), see [7, Th. 5]. Some have a cocompact automorphism group, with a global fixed vertex at infinity, see [71, §7]. The existence of such a fixed point prevents that group from being unimodular (see [22, Th. M]), and hence cannot contain any lattice. The known exotic \tilde{A}_2 -buildings relevant to Theorem 1.1 are those endowed with a proper cocompact action of a discrete group. They are constructed as universal covers of finite complexes having finite projective planes as residues, or of suitably defined finite complexes of groups. The constructions that we are aware of are the following (the **order** of a locally finite \tilde{A}_2 -building is the common order of its **residue planes**, i.e. the finite projective planes appearing as its rank 2 residues).

- In [69, §3.1] and [56, Th. 2.5], Tits and Ronan construct four \tilde{A}_2 -buildings of order 2 with cocompact lattices. The lattices act regularly on the set of chambers of their building. It is shown in [68, §3.1] and [46] that two of those are Bruhat–Tits buildings with arithmetic lattices, while the other two are exotic (see also [43, §3.6]).
- In [69, §3.1] and [68, §3.2], Tits constructs 45 \tilde{A}_2 -buildings of order 8 with cocompact lattices acting regularly on the set of chambers of their building. It is asserted that 43 of them are exotic.
- In [56, §3] and [43, §C.3], Ronan and Kantor describe a framework providing in particular cocompact lattices of \tilde{A}_2 -buildings acting sharply transitively on the panels of each given type. Such lattices are called **panel-regular**.
- In [26], Cartwright–Mantero–Steger–Zappa construct 65 exotic \tilde{A}_2 -buildings of order 3 with cocompact lattices. In those examples, the lattices act regularly on the set of vertices of their building; in particular the types of vertices are permuted cyclically.
- In [6, §3], Barré construct an exotic \tilde{A}_2 -building of order 2 with a cocompact lattice. The full automorphism group of that building is discrete, and has exactly two orbits of vertices.
- In [33], Essert provides presentations of arbitrary panel-regular lattices in \tilde{A}_2 -buildings. When the building is of order 2, there are exactly two inequivalent presentations of such groups. One of them gives rise to an arithmetic group, the other is exotic (see Remark D.8 and Lemma D.9 below). The \tilde{A}_2 -buildings of order 3 and 5 admitting a panel-regular lattice have recently been classified by S. Witzel (see [74]), who determined

which are exotic and described their full automorphism group. We show in Appendix D how to use Theorem 1.1 to construct, from a given panel-regular lattice in an exotic \tilde{A}_2 of order p , an infinite family of panel-regular lattice in an exotic \tilde{A}_2 of order p^n . In particular we obtain cocompact lattices in exotic \tilde{A}_2 -buildings of arbitrarily large order.

- In all of the previous examples, the residue planes are always Desarguesian. Recently, N. Radu [54] constructed an example of an exotic \tilde{A}_2 -building of order 9, with a vertex-transitive discrete automorphism group and with non-Desarguesian residues (isomorphic to the Hughes plane of order 9).

Remark that in all those examples, the lattices are cocompact: no exotic example with a non-uniform lattice is known.

Exotic buildings of type \tilde{C}_2 and \tilde{G}_2 have been less investigated, but some constructions are known, see [42] and [33]. In fact, the most fascinating example of an exotic 2-dimensional affine building with a cocompact lattice is probably the building of type \tilde{G}_2 constructed by Kantor [42] as the universal cover of a simplicial complex associated with the finite sporadic simple group of Lyons (see also [68, §3.5]). The corresponding building is exotic (an assertion attributed to Tits, without proof, in [42]). This can be deduced from the fact that the full automorphism group of that building is discrete, a fact established by Grüniger [38, Cor. 3.3].

1.4 Structure of the paper

The next two sections are of introductory nature and essentially contain no new results. The next section, §2, serves as a concise introduction to \tilde{A}_2 -buildings. In particular we recall in this section the notions of *wall trees* and *panel trees* which are of great importance in this work. In §3 we recall the notion of the *projectivity group* and explain how it acts on panel trees. In subsection §3.3 we discuss linear representations and algebraicity of groups acting on trees. In §4 we use the results of the previous section in order to study linear representations of the projectivity group. The main result of this section is Theorem 4.7 which roughly states that the projectivity group of a building admits a linear representation if and only if the building is Bruhat–Tits. This result may be of independent interest. The proof of Theorem 1.5 will be carried eventually by a reduction to this theorem.

The next three sections, §5, §6 and §7 are devoted to the construction and the study of some new spaces, the most important of which are the *the space of marked flats* and *space of (restricted) marked wall trees*. We define these as topological spaces in §5 and introduce their measured structure in §6. Finally, in §7, we prove ergodicity results for these spaces. The ergodicity of the singular Cartan flow, Theorem 7.3, is the technical heart of this paper.

In the last section of the paper, §8, we combine Theorem 7.3 with *the theory of algebraic gates* in order to reduce the proof of Theorem 1.5 to Theorem 4.7. We then proceed and prove Theorem 1.1.

The paper has four appendices. The first two are actually essential ingredients in the proof of the main theorems of the paper, which are presented as appendices due to there independent characters. The first is §A which is devoted to the proof of Theorem A.1. This theorem is the main ingredient in the reduction of Theorem 1.1 to Theorem 1.5. In §B we review the classical Hopf argument in an abstract setting. In particular we prove Theorem B.1, which is essential in the proofs given in §7.

The last two appendices contain supplementary results. In §C we consider Galois lattices. The main result in this section is Theorem C.3 which essentially states that Galois lattices in

higher-rank groups are non-linear. In the last appendix, §D, we use the results of the paper to construct a concrete infinite family of exotic \tilde{A}_2 -lattices. This is done in Corollary D.10.

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2 Triangular buildings and their boundaries

In this section we recall some basic facts on \tilde{A}_2 -buildings and their boundaries. In §2.1 we give a general introduction to \tilde{A}_2 -buildings and in §2.2 we discuss their boundaries. In §2.3 we discuss the important notions of *wall trees* and *panel trees*. Most of the results we will state here are well known and we will not provide proofs in these cases. Standard references to the subject are [2], [35] and [58].

2.1 Buildings of type \tilde{A}_2

Let us start by recalling the definition of a building of type \tilde{A}_2 (or \tilde{A}_2 -building for short). We denote by Σ a **model apartment** : it is the triangular complex obtained by tessellation of the Euclidean plane \mathbf{R}^2 by equilateral triangle. A **wall** in Σ is a line in Σ which is a union of edges. We fix a vertex $0 \in \Sigma$.

Definition 2.1. A **building** of type \tilde{A}_2 is a triangular complex satisfying the two following conditions, in which we call **apartment** a subcomplex of X isomorphic to Σ :

- Any two simplices of X are contained in an apartment,
- If F and F' are two apartments, there exists a simplicial isomorphism $F \rightarrow F'$ which fixes pointwise $F \cap F'$.

The 2-dimensional simplices of an \tilde{A}_2 -building are called **chambers**. The space X is endowed with a natural metric, defined as follows: the distance between two points x and y is the distance between x and y calculated in any apartment containing x and y . This is well-defined, and furthermore gives a CAT(0) metric on X . Apartments are also called **flats**, because in the CAT(0)-metric on X , the apartments are maximal subspaces isometric to a Euclidean space.

An \tilde{A}_2 -building X is equipped with a coloring $X^{(0)} \rightarrow \{0, 1, 2\}$ of its vertex set such that each chamber has exactly one vertex of each color. The color of a vertex is called its **type**.

We assume throughout that the building X is **thick**, i.e. there is more than two chambers adjacent to each edge.

It is worth noting that the local geometry is forced by the geometry of the building. For a start, the building is **regular**: the cardinality of the set of chambers adjacent to an edge is constant (see for example [50, Theorem 2.4]). When that number is finite, we define the **order** of the \tilde{A}_2 -building X as the number q such that the number of chambers adjacent to each edge equals $q + 1$. Since we assumed X to be thick we have $q > 1$.

Furthermore, the link at a vertex is always a (finite) projective plane, as is the boundary of the building, as we explain below. For the convenience of the reader, we recall the definition¹:

Definition 2.2. A **projective plane** is a bipartite graph, with two types of vertices (called **points** and **lines**), such that

- Any two lines are adjacent to a unique point,
- Any two points are adjacent to a unique line.
- There exist four points, no three of which are adjacent to a common line.

Projective planes are also known as generalized triangles, or buildings of type A_2 .

Another striking feature of \tilde{A}_2 (or more general affine) buildings is the following fact:

Theorem 2.3. *Let Ω be a subset of X which is either convex or of non-empty interior. If Ω is isometric to a subset of \mathbf{R}^2 , then there exists an apartment F such that $\Omega \subset F$.*

An important class of subsets of X to which we will apply this theorem is the following:

Definition 2.4. A **sector** based at 0 in Σ is a connected component of the complement in Σ of the union of the walls passing through 0.

A sector of X is a subset of X which is isometric to a sector in Σ .

2.2 Boundaries of buildings

Let X be a building of type \tilde{A}_2 . Since X is a CAT(0) space, its ideal boundary ∂X is naturally endowed with the structure of a CAT(1) space. The combinatorial nature of X is such that ∂X inherits a much finer structure, namely that of a compact projective plane, equipped with a family of natural measures.

2.2.1 The projective plane at infinity

A boundary point $v \in \partial X$ is called a **vertex at infinity** if it is the endpoint of a geodesic ray parallel to a wall in X . Two sectors S and S' in X are **equivalent** if their intersection contains a sector; equivalently their ideal boundaries ∂S and $\partial S'$ coincide. An equivalence class of sectors (or, equivalently, the ideal boundary of a sector) is called a **Weyl chamber**.

Two Weyl chambers are called **adjacent** if they contain a common vertex at infinity. This happens if and only if they are represented by two sectors whose intersection contains a geodesic ray. We let Δ denote the graph whose vertices are the vertices at infinity, such that two vertices define an edge if the corresponding vertices at infinity lie in a common Weyl chamber. The set of edges of Δ is denoted by $\text{Ch}(\Delta)$. The ideal boundary ∂X may be viewed as a CAT(1) metric realization of the graph Δ .

We can finally state the fact alluded to above.

Theorem 2.5. *The graph Δ is a projective plane.*

¹Strictly speaking, the definition we give is the definition of the incidence graph of a projective plane; the identification of a projective plane and its incidence graph implicit throughout this paper should not cause any confusion.

A key point in the proof of this theorem is to prove that two chambers are always in some apartment at infinity. An intermediate step is the following lemma, which is also useful for us:

Lemma 2.6. *Let x be a vertex in X , and $C \in \text{Ch}(\Delta)$. There exists a unique sector based at x in the equivalence class of C . We denote this sector by $Q(x, C)$.*

Two chambers in Δ are called **opposite** if they are at maximal distance (*i.e.* at distance three). Similarly, two vertices are called opposite if they are at maximal distance.

By convexity of apartments, the convex hull of two chambers is contained in any apartment containing both of them. This explains the following.

Lemma 2.7. *Two opposite chambers are contained in a unique apartment.*

We note for further use the following standard fact [72, Proposition 29.50]:

Lemma 2.8. *For every pair of chambers $x, y \in \text{Ch}(\Delta)$, there exists a chamber $z \in \text{Ch}(\Delta)$ which is opposite to both x and y .*

2.2.2 Topology on Δ

The projective plane Δ , seen as the boundary of X , is endowed with a natural topology: indeed Δ is naturally endowed with the structure of a compact projective plane. One way to topologize Δ consists in observing that the vertex set of Δ is a closed subset of the ideal boundary ∂X endowed with the cone topology. Moreover, viewing each Weyl chamber as a subset of ∂X , it turns out that the set of barycenters of Weyl chambers is closed in the cone topology. In this way the vertex-set and edge-set of Δ are naturally endowed with a topology, which is compact and totally disconnected (see [23, Prop. 3.5]). A basis of open sets of the topology on $\text{Ch}(\Delta)$ is provided by the sets of the form $\Omega_x(y) := \{C \in \text{Ch}(\Delta) \mid y \in Q(x, C)\}$, where x and y are vertices of X .

We note the following fact for further use.

Lemma 2.9. *Let C be a chamber. The set of chambers which are opposite to C is an open subset of $\text{Ch}(\Delta)$.*

Proof. Fix an apartment A containing C , and let C' be the chamber in A opposite to C . Let O be the set of chambers opposite to C . Then O is the union of sets of the form $\Omega_x(y)$, where x is a vertex in A and y is a vertex in the sector $Q(x, C')$. \square

2.3 Wall trees and panel trees

An important tool in our study of triangular buildings are **wall trees** and **panel trees**. These closely related notions were introduced by Tits in his study of the structure and classification of affine buildings. A good reference is [73, Chapter 10 and 11].

Let us start by defining wall trees. These trees are defined as equivalence classes of parallel lines. A geodesic line l is said **singular** if it is parallel to a wall in some (hence any) apartment containing it, otherwise it is called **regular**.

Definition 2.10. Two geodesic lines ℓ and ℓ' are called **parallel** if they are contained in a common apartment, and are parallel in this apartment. The **distance** $d(\ell, \ell')$ between parallel lines is the Euclidean distance between them in any apartment containing them.

Parallelism is an equivalence relation. Furthermore, the distance between parallel lines turns indeed each class into a metric space.

For regular lines, this space is just isometric to \mathbf{R} . However, these spaces turn out to be interesting for singular lines [73, Proposition 10.14 and Corollary 10.16]:

Proposition 2.11. *Let ℓ be a singular line, and T_ℓ be the set of lines parallel to ℓ . Then T_ℓ is a thick tree. If X is locally finite of order q , then T_ℓ is regular of degree $q + 1$.*

The tree T_ℓ defined above is called a **wall tree**. Note that every geodesic line parallel to ℓ has the same two endpoints at infinity as ℓ . These two endpoints are by definition vertices (or panels) of the building at infinity Δ .

Now we turn to the definition of a panel tree. Let v be a vertex of Δ . Then v is represented by a class of geodesic rays, two of them being equivalent if they are parallel after a certain time.

Definition 2.12. Two geodesic rays are said **asymptotic** if their intersection contains a ray.

If two rays $r, r' : [0, +\infty) \rightarrow X$ are parallel, their **distance** is defined as

$$\inf_s \lim_{t \rightarrow +\infty} d(r(t+s), r'(t)).$$

It is clear that this distance only depends on the asymptotic class of r and r' . It is possible to check that we have indeed a distance on the set of asymptotic classes of rays in the class of v .

Proposition 2.13. *The metric space, denoted T_v , of asymptotic classes of rays in the class of v , is a thick tree. If X is locally finite of order q , then T_v is regular of degree $q + 1$.*

This proposition can be found in [73, Corollary 11.18]. The tree T_v is called the **panel tree** associated to v .

Panel trees are related to wall trees via canonical maps [73, Proposition 11.16]:

Proposition 2.14. *Let ℓ be a singular geodesic line, and v be an endpoint of ℓ . Then the map $T_\ell \rightarrow T_v$ which associates to a line parallel to ℓ its asymptotic class is an isometry.*

The set of ends of a panel tree is also well understood. Any ray in T_v can be lifted to a sequence of adjacent parallel geodesic rays in some apartment F containing v . To such a sequence one can associate the chamber, in the boundary of F , which contains v and goes in the direction of the sequence of rays. This explains the construction of the bijection in the following proposition [73, Proposition 11.22]:

Proposition 2.15. *Let v be a vertex in Δ . There is a canonical $\text{Aut}(X)_v$ -equivariant bijection between the set of ends of the panel tree T_v and the set $\text{Ch}(v)$ of chambers of Δ containing v .*

3 Structure of the projectivity group

The projectivity group is an important invariant of the building at infinity Δ , and plays a crucial role in our proofs. In this section we will define it and discuss its properties. In the next section, §4, we will study its linear representations. First, in §3.1 we will define the projectivity group in the context of a general projective plane, and then, in §3.2, in the restricted setting of an affine building. In the latter setting, we will explain how the projectivity group is acting on a panel tree. The last subsection, §3.3, is devoted to the general study of linear representations of groups acting on trees. This is a preparation for §4.

3.1 Projectivity groups of projective planes

In this section, we recall some basic terminology on projectivity groups, and record important known results that will be needed in the sequel.

Let Δ be a thick building of type A_2 (i.e. Δ is a projective plane). If v is a vertex of Δ , we denote by $\text{Ch}(v)$ the set of chambers of Δ adjacent to v .

Let v and v' be opposite vertices of Δ . The **combinatorial projection** to v is the map from Δ to $\text{Ch}(v)$ which associates to $C \in \Delta$ the unique chamber of $\text{Ch}(v)$ which is at minimal distance from C . The fact that we chose v and v' opposite implies that this projection, restricted to $\text{Ch}(v')$, is an involutory bijection.

Given a sequence (v_0, \dots, v_k) of vertices such that v_i is opposite v_{i-1} and v_{i+1} for all $i \in \{1, \dots, k-1\}$, we obtain a bijection of $\text{Ch}(v_0)$ to $\text{Ch}(v_k)$ by composing the successive projection maps from $\text{Ch}(v_{i-1})$ to $\text{Ch}(v_i)$. That bijection is called a **perspectivity**; it is denoted by $[v_0; v_1; \dots; v_k]$. In the special case when $v_0 = v_k$, it is called a **projectivity** at v_0 . The collection of all projectivities at a vertex v_0 is denoted by $\Pi(v_0)$; it is a permutation group of the set $\text{Ch}(v_0)$, called the **projectivity group of Δ at v** and denoted by $(\Pi(v_0), \text{Ch}(v_0))$. The isomorphism type of the permutation group $(\Pi(v_0), \text{Ch}(v_0))$ does not depend on the choice of the vertex v_0 . Indeed, given another vertex v , there exists a perspectivity from $\text{Ch}(v_0)$ to $\text{Ch}(v)$ (because Δ is thick, see Lemma 2.8), which conjugates $(\Pi(v_0), \text{Ch}(v_0))$ onto $(\Pi(v), \text{Ch}(v))$. For that reason, it makes sense to define **the projectivity group** of the building Δ as any representative (Π, Ω) of the isomorphism class of the projectivity group $(\Pi(v), \text{Ch}(v))$ at a vertex v .

Proposition 3.1. *The projectivity group (Π, Ω) of a thick building of type A_2 is 3-transitive.*

Proof. See Proposition 2.4 in [17]. □

We remark that projectivity groups can be defined for arbitrary generalized polygons; a result of N. Knarr [44, Lemma 1.2] ensures that they are always 2-transitive.

The projectivity group (Π, Ω) is an important invariant of a projective plane Δ . For example, it is known that (Π, Ω) is sharply 3-transitive if and only if Δ is a projective plane over a commutative field, see Theorem 2.5 in [17]. We will need an important variant of that fact. In order to state it, we recall that a 2-transitive permutation group (Π, Ω) is called a **Moufang set** if the point stabilizer Π_x contains a normal subgroup U_x acting sharply transitively on $\Omega \setminus \{x\}$. Any conjugate of U_x in Π is then called a **root group**.

Theorem 3.2. *Let Δ be a thick building of type A_2 . Then Δ is Moufang if and only if the projectivity group (Π, Ω) is a Moufang set.*

Proof. See the main result from [60]. □

We will also need the following elementary criterion, ensuring that a projectivity of a very specific form is non-trivial.

Lemma 3.3. *Let Δ be a thick building of type A_2 . Let C, C_0, C_1, C_2, C_3 be chambers such that $\Sigma_{i,j} = C_i \cup C \cup C_j$ is a half-apartment of Δ for all $(i, j) \in \{0, 2\} \times \{1, 3\}$.*

For all $i \pmod 4$, let v_i be the vertex of C_i which lies on the boundary of $\Sigma_{i,i+1}$. If $C_0 \neq C_2$ and $C_1 \neq C_3$, then the projectivity $[v_0; v_1; v_2; v_3; v_0]$ is non-trivial: indeed, its only fixed point in $\text{Ch}(v_0)$ is C_0 .

Proof. Let $C'_0 \in \text{Ch}(v_0)$ be any chamber different from C_0 . We claim that the projectivity $g = [v_0; v_1; v_2; v_3; v_0]$ does not fix C'_0 . Indeed, we define inductively, for each $i = 0, 1, 2, 3$, the chamber C'_{i+1} as the projection of C'_i to $\text{Ch}(v_{i+1})$, and D_i as the unique chamber which is adjacent to both C'_i and C'_{i+1} . It then follows that for all $i = 0, 1, 2$, the chamber D_i is adjacent to D_{i+1} . Moreover D_3 is adjacent to C'_4 .

By definition, we have $g(C'_0) = C'_4$. Therefore, if $g(C'_0) = C'_0$, then D_3 is adjacent to D_0 , so that $(D_0, D_1, D_2, D_3, D_0)$ forms a closed gallery of length 4. Therefore, we must have $D_0 = D_1$ or $D_1 = D_2$, which forces respectively $C_0 = C_2$ or $C_1 = C_3$. \square

3.2 Projectivity groups of Euclidean buildings

Now, let us go back to the situation when X is a thick Euclidean building of type \tilde{A}_2 , and Δ is the set of chambers of its building at infinity. In that particular case, the projectivity group (Π, Ω) of Δ is more than just a permutation group of a set Ω . Indeed, to each vertex $v \in \Delta$, one associates the panel-tree T_v , as in Proposition 2.13.

Let v and v' be two opposite vertices of different types in Δ , and let ℓ be a geodesic line in X joining these two vertices. By Proposition 2.14, the panel trees T_v and $T_{v'}$ are both canonically isomorphic to the wall tree T_ℓ . Hence we get a canonical involutory isomorphism between the panel-trees T_v and $T_{v'}$. From that observation, it follows that any perspectivity of $\text{Ch}(v)$ to $\text{Ch}(v')$ defines an isomorphism of panel-trees from T_v to $T_{v'}$, and that the projectivity group at v acts by automorphisms on the panel-tree T_v . We define the **projectivity group of X at v** as $(\Pi(v), T_v)$ by viewing the usual projectivity group $\Pi(v)$ as a subgroup of $\text{Aut}(T_v)$. Thus the projectivity group of Δ at v is nothing but $(\Pi(v), \partial T_v)$, since the set $\text{Ch}(v)$ is canonically isomorphic to the set of ends of the panel tree T_v by Proposition 2.15. Note that using isomorphisms as described above, all wall trees and all panel trees are mutually isomorphic (though not canonically).

Definition 3.4. We fix once and for all a tree T which is isomorphic to all panel and wall trees in X and call it **the model tree** of X . By choosing an actual identification of T with a panel tree T_v and pulling back the projectivity group $\Pi(v)$ we define Π , the **model projectivity group** of X .

Lemma 3.5. *T is a regular tree which is locally finite (of degree $q+1$) if and only if X is locally finite (of order q). The projectivity group Π acts 3-transitively on ∂T and vertex-transitively on T .*

Proof. It is easy to see that T is locally finite if and only if X is locally finite. The 3-transitivity of Π on ∂T follows from Proposition 3.1. Π is vertex-transitive on T , since every vertex can be uniquely determined by a triple of distinct ends. The regularity of T follows. \square

3.3 Algebraic groups and locally finite trees

In this section we recall some general results on groups acting on trees that we need in our analysis of the group of projectivities of an \tilde{A}_2 -building.

3.3.1 Linear simple locally compact groups are algebraic

A locally compact group is called **linear** if it has a continuous, faithful, finite-dimensional representation over a local field.

Theorem 3.6. *Let G be a t.d.l.c. group such that the intersection G^+ of all non-trivial closed normal subgroups of G is compactly generated, topologically simple and non-discrete, and that the quotient G/G^+ is compact. If G has a linear open subgroup, then the following assertions hold.*

- (i) G^+ is algebraic: indeed G^+ is isomorphic to $H(k)/Z$, where k is a local field, H is an absolutely simple, simply connected algebraic group defined and isotropic over k , and Z is the center of $H(k)$.
- (ii) G/G^+ is virtually abelian.

Proof. Assertion (i) follows from Corollary 1.4 from [24]. Assertion (ii) follows from Theorem 1.1 in [24] and the fact that the compact open subgroups of G^+ are not solvable (indeed they are Zariski-dense in a simple algebraic group). \square

3.3.2 Boundary-transitive automorphism groups of trees

The following result, due to Burger–Mozes [15], provides a natural set-up where all hypotheses of Theorem 3.6 other than the linearity are satisfied.

Theorem 3.7. *Let T be a thick locally finite tree and $G \leq \text{Aut}(T)$ be a closed subgroup acting 2-transitively on the set of ends ∂T .*

Then the intersection G^+ of all non-trivial closed normal subgroups of G is compactly generated, topologically simple and non-discrete. It acts 2-transitively on ∂T , and the quotient G/G^+ is compact. In particular, G is unimodular.

Proof. See Proposition 3.1.2 from [15] and Theorem 2.2 from [21]. For the unimodularity of G , observe that G^+ must be in the kernel of the modular character of G . Thus the modular character of G factors through the quotient G/G^+ , which is compact. Hence G is indeed unimodular. \square

An automorphism of a tree T is called **horocyclic** if it fixes pointwise a horosphere of T . Combining the previous two statements, we obtain a subsidiary result which is a key tool to the proof of our characterization.

Proposition 3.8. *Let T be a thick locally finite tree and $G \leq \text{Aut}(T)$ be a closed subgroup acting 2-transitively on the set of ends ∂T . If G has a linear open subgroup, then the following assertions hold.*

- (i) *The group G^+ , defined as in Theorem 3.7, is isomorphic to $H(k)/Z$, where k is a local field, H is an absolutely simple, simply connected algebraic k -group of k -rank one, and Z is the center of $H(k)$.*
- (ii) *$(G, \partial T)$ is a Moufang set, whose root groups are the unipotent radicals of proper k -parabolic subgroups of G^+ .*
- (iii) *Each element $u \in G$ which is a horocyclic automorphism of T belongs to some root group.*
- (iv) *The quotient G/G^+ is virtually abelian.*

Proof. By Theorems 3.6 and 3.7, we know that G^+ , defined as in Theorem 3.7, is isomorphic to $H(k)/Z$, where k is a local field, H is an absolutely simple, simply connected algebraic k -group of k -rank > 0 . Since G^+ acts properly and cocompactly on a thick locally finite tree, its k -rank must be 1. This proves (i).

It is well known, and easy to see using maximal compact subgroups, that G^+ has, up to isomorphism, a unique proper cocompact action on a thick locally finite tree (see e.g. Lemma 2.6(viii) from [20] for a proof). Thus T is isomorphic to the Bruhat–Tits tree of G^+ . It follows that $(G^+, \partial T)$ is a Moufang set, whose root groups are the unipotent radicals of proper k -parabolic subgroups of G^+ . By [53, Lemma 2.4], those unipotent radicals also coincide with the contraction groups of hyperbolic elements of G^+ , where the **contraction group** of an element h is defined as

$$\text{con}(h) = \{g \in G^+ \mid h^n g h^{-n} \rightarrow 1\}.$$

Since G^+ is normal in G , it follows that the root groups are permuted by conjugation under G . Therefore, the permutation group $(G, \partial T)$ is also a Moufang set. This proves (ii).

Finally, let $u \in G$ be a horocyclic element, and let $e \in \partial T$ be the center of the horosphere of T fixed by u . Given any hyperbolic element $h \in G$ whose repelling fixed point is e , we have $u \in \text{con}(h)$. We choose such a hyperbolic h in G^+ (this is possible because G^+ is 2-transitive on ∂T). The image of $\text{con}(h)$ in G/G^+ is then trivial, so that $\text{con}(h) \leq G^+$. As observed above, it follows that $\text{con}(h)$ is a root group. This proves (iii). The assertion (iv) is clear from Theorems 3.6 and 3.7. \square

4 Structure and representations of the group of projectivities

The goal of this section is to prove Theorem 4.7 which roughly states that the projectivity group is linear if and only if the building X is a Bruhat–Tits building. The proof of this theorem will be carried in §4.2. It will rely on Proposition 4.1 below, which proof will be our main concern in §4.1.

4.1 Existence of horocyclic elements

This subsection will be devoted to the proof of the following technical result.

Proposition 4.1. *Let X be a thick Euclidean building of type \tilde{A}_2 . Then the projectivity group contains a non-trivial horocyclic automorphism of the panel-tree.*

We need to collect a number of subsidiary facts, assuming throughout that X is a thick Euclidean building of type \tilde{A}_2 ; the proof of Proposition 4.1 is deferred to the end of the section.

Lemma 4.2. *Let e_0, e_1, e_2, e_3 be vertices of Δ , and $D_0, D_1, D_2, D_3 \subset X$ be geodesic lines such that the endpoints of D_i are $\{e_i, e_{i+1}\}$ for all i , where the indices are written modulo 4.*

If $D_0 \cap D_3$ is non-empty, then the projectivity $[e_0; e_1; e_2; e_3; e_0]$ fixes the point of the panel-tree T_{e_0} represented by D_0 .

Proof. Let $x \in T_{e_0}$ be the point represented by D_0 . Its image under the projectivity $g = [e_0; e_1; e_2; e_3; e_0]$ is the point of T_{e_0} represented by D_3 . Since the intersection $D_0 \cap D_3$ is non-empty, it contains a geodesic ray tending to e_0 . Thus g fixes x as claimed. \square

Lemma 4.3. *Let e_0, e_1, e_2 be vertices of Δ , and $D_0, D_1 \subset X$ be geodesic lines such that the endpoints of D_i are $\{e_i, e_{i+1}\}$ for $i = 0, 1$.*

Suppose that the point $x \in T_{e_0}$ represented by D_0 is a vertex and let $x' \in T_{e_0}$ be a vertex adjacent to x . Let D'_0 be the geodesic line parallel to D_0 and representing x' . Let also D'_1 be the geodesic line parallel to D_1 and representing the same point of T_{e_1} as D'_0 .

Let c be a chamber of X with vertices u, v, w . If u and v are both contained in $D_0 \cap D_1$ and if w is contained in D'_0 , then $w \in D'_1$.

Proof. We refer to Figure 1. Upon exchanging u and v , we may and shall assume that u

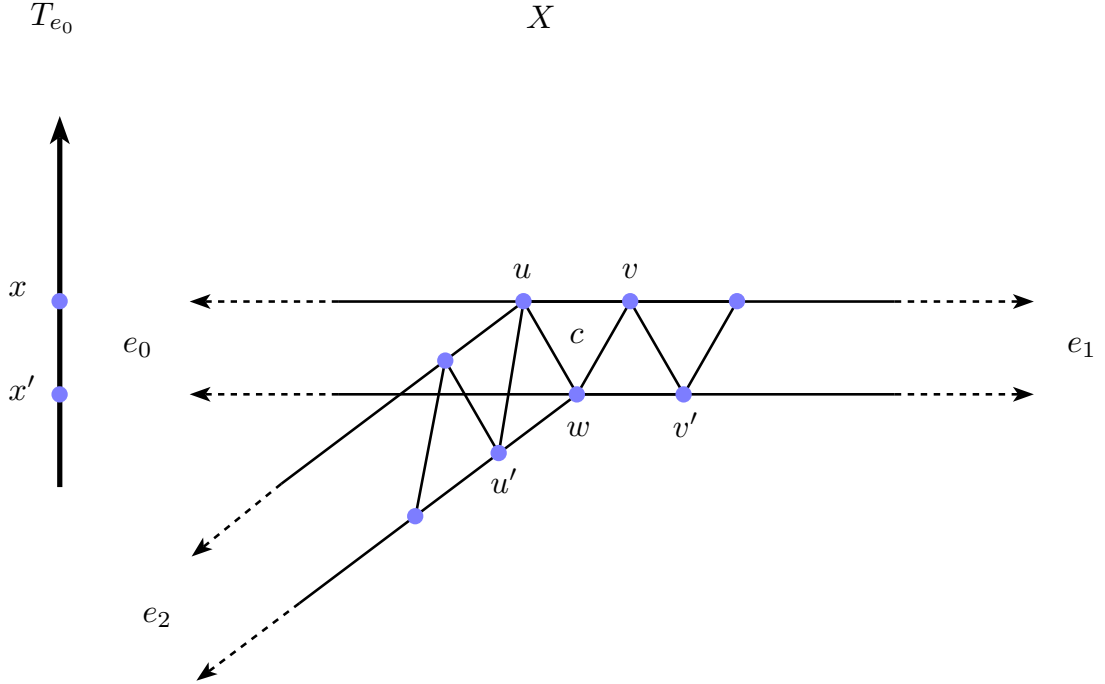


Figure 1: Illustration of Lemma 4.3

belongs to the geodesic ray $[v, e_0)$. Therefore u also belongs to the geodesic ray $[v, e_2)$ since $u, v \in D_1$.

Let u' (resp. v') be the first vertex different from w on the geodesic ray $[w, e_2)$ (resp. $[w, e_1)$). We claim that u' is opposite v' in the residue $\text{Res}(w)$ of the vertex w .

To establish the claim, we first observe that u' is adjacent to u . Indeed, by hypothesis the first vertex different from v on the geodesic segment $[v, e_2)$ is u . Since the line D_1 is singular, it is a wall in any apartment that contains it. Since the chamber $c \in \text{Ch}(X)$ has a panel on D_1 , we may find an apartment containing both the line D_1 and the chamber c . In particular, that apartment contains the geodesic rays $[v, e_2)$ and $[w, e_2)$, which are parallel. It follows that u is adjacent to w as desired.

This observation implies that the vertices u' and v have the same type. Hence u' and v' have different types. Therefore, if u' and v' are not opposite in $\text{Res}(w)$, then they are adjacent. But the unique vertex in $\text{Res}(w)$ which is adjacent to both u and v' is v . Thus,

if the claim fails, then we have $u' = v$. This implies that the first vertex different from w on $[w, e_2)$ is v . Therefore the first vertex different from v on $[v, e_2)$ is opposite w in $\text{Res}(v)$. On the other hand, as seen above, the first vertex different from v on $[v, e_2)$ is u . Thus u is opposite w in $\text{Res}(v)$. This is absurd since u is adjacent to w . The claim stands proven.

The claim implies that the union of the geodesic rays $(e_1, w] \cup [w, e_2)$ is a geodesic line intersecting D'_0 in the geodesic ray $[w, e_1)$. Therefore that line must coincide with D'_1 , which confirms that $w \in D'_1$. \square

Lemma 4.4. *Let e_0, e_1, e_2, e_3 be vertices of Δ , and $D_0, D_1, D_2, D_3 \subset X$ be geodesic lines such that the endpoints of D_i are $\{e_i, e_{i+1}\}$ for all $i \bmod 4$.*

Suppose that the point $x \in T_{e_0}$ represented by D_0 is a vertex and let $x' \in T_{e_0}$ be a vertex adjacent to x . Let D'_0 be the geodesic line parallel to D_0 and representing x' . For $i > 0$, define inductively D'_i as the geodesic line parallel to D_i and representing the same point of T_{e_i} as D'_{i-1} .

For each $n > 0$, if $D_0 \cap D_1 \cap D_2 \cap D_3$ is a geodesic segment of length $\geq n$, then $D'_0 \cap D'_1 \cap D'_2 \cap D'_3$ is a geodesic segment of length $\geq n - 1$.

Proof. Since the line D_0 represents a vertex of T_{e_0} , it is singular, i.e. it lies on the boundary of a half-apartment, and thus entirely consists of vertices and edges of X . Let (v_0, v_1, \dots, v_m) be the geodesic segment (where v_i is a vertex) which is the intersection $D_0 \cap D_1 \cap D_2 \cap D_3$, so that $m \geq n$ by hypothesis.

For all $i \in \{0, m-1\}$, let v'_i be the unique vertex belonging to D'_0 which is adjacent to both v_i and v_{i+1} . By invoking Lemma 4.3 three times successively, we see that v'_i is contained in D'_1, D'_2 and D'_3 . This proves that the segment $[v'_0, v'_{m-1}]$, which is of length $m-1 \geq n-1$, is contained in $D'_0 \cap D'_1 \cap D'_2 \cap D'_3$. \square

Lemma 4.5. *Let $e_0, e_1, e_2, e_3 \in X(\infty)$ be vertices, and $D_0, D_1, D_2, D_3 \subset X$ be geodesic lines such that the endpoints of D_i are $\{e_i, e_{i+1}\}$ for all $i \bmod 4$. Suppose that the point $x \in T_{e_0}$ represented by D_0 is a vertex.*

If $D_0 \cap D_1 \cap D_2 \cap D_3$ is a geodesic segment of length $\geq n$, then the projectivity $[e_0; e_1; e_2; e_3; e_0]$ fixes pointwise the ball of radius n around x .

Proof. Let x' be a vertex at distance $r \leq n$ from x . Let D'_0 be the geodesic line parallel to D_0 and representing x' . For $i > 0$, define inductively D'_i as the geodesic line parallel to D_i and representing the same point of T_{e_i} as D'_{i-1} .

An immediate induction on $r = d(x, x')$, using Lemma 4.4, shows that $D'_0 \cap D'_1 \cap D'_2 \cap D'_3$ is a geodesic segment of length $\geq n - r$. By Lemma 4.2, this implies that x' is fixed by $[e_0; e_1; e_2; e_3; e_0]$. \square

Lemma 4.6. *Let $v \in X$ be a vertex. Let Q, Q_0, Q_1, Q_2, Q_3 be sectors based at v , such that $A_{i,j} = Q_i \cup Q \cup Q_j$ is a half-apartment of X for all $(i, j) \in \{0, 2\} \times \{1, 3\}$. For all $i \bmod 4$, let D_i be the boundary line of $A_{i,i+1}$, and let e_i be the common endpoint of D_{i-1} and D_i .*

Then the projectivity $[e_0; e_1; e_2; e_3; e_0]$ is a horocyclic automorphism of the tree T_{e_0} .

Proof. We refer to Figure 2.

Let $x_0 \in T_{e_0}$ be the vertex represented by D_0 , and let (x_0, x_1, \dots) be the geodesic ray of T_{e_0} represented by the sector Q_0 .

We claim that the projectivity $u = [e_0; e_1; e_2; e_3; e_0]$ fixes pointwise the ball of radius n around x_n , so that u is indeed horocyclic.

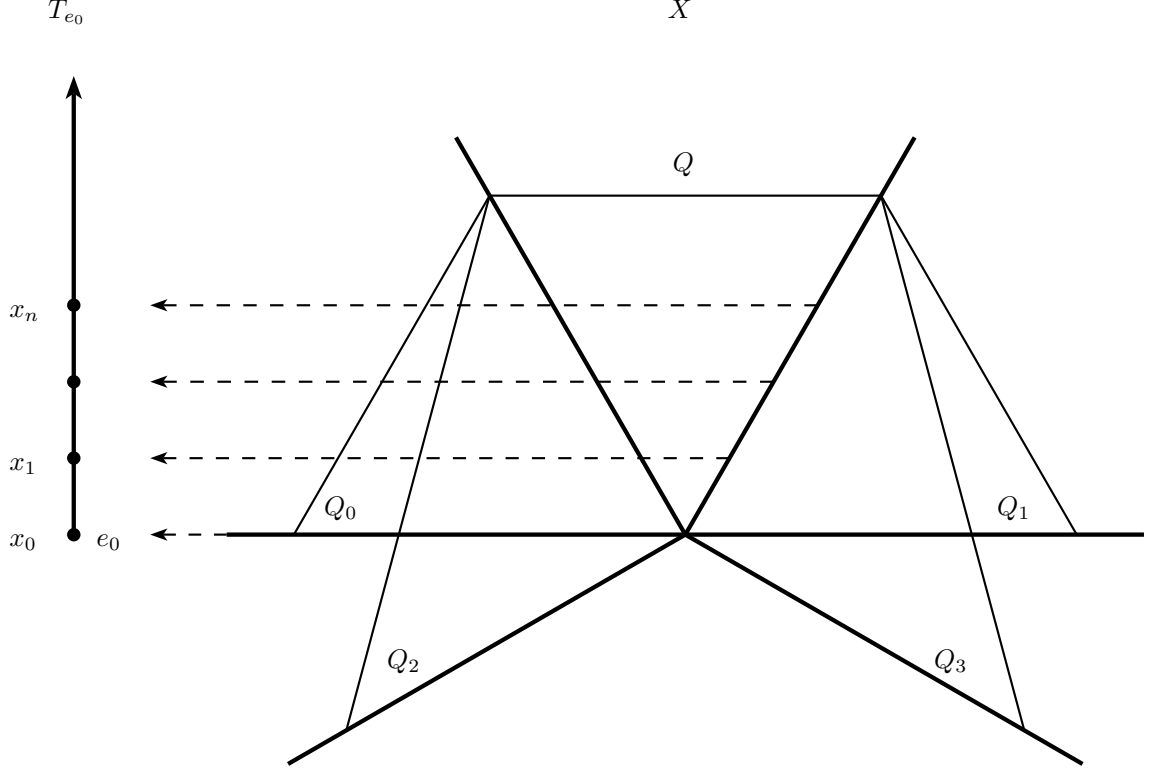


Figure 2: Illustration of Lemma 4.6

To see this, let D'_0 be the geodesic line parallel to D_0 and representing x_n . For $i > 0$, define inductively D'_i as the geodesic line parallel to D_i and representing the same point of T_{e_i} as D'_{i-1} .

By construction the line D'_i is contained in the half-apartment $A_{i,i+1}$. Therefore, it intersects the sector Q in a segment of length n . In particular the intersection $D'_0 \cap D'_1 \cap D'_2 \cap D'_3$ is a geodesic segment of length $\geq n$. The claim then follows from Lemma 4.5. \square

Proof of Proposition 4.1. Let $v \in X$ be a vertex. Let Q, Q_0, Q_1, Q_2, Q_3 be sectors based at v , such that $A_{i,j} = Q_i \cup Q \cup Q_j$ is a half-apartment of X for all $(i, j) \in \{0, 2\} \times \{1, 3\}$. Since X is thick, we may choose those sectors such that $Q_i \cap Q_{i+2}$ is a geodesic ray, for all $i = 0, 1$. In particular, denoting the boundary chamber of Q_i by $C_i \in \text{Ch}(X(\infty))$, we have $C_0 \neq C_2$ and $C_1 \neq C_3$.

For all $i \bmod 4$, let D_i be the boundary line of $A_{i,i+1}$, and let e_i be the common endpoint of D_{i-1} and D_i .

By Lemma 4.6, the projectivity $u = [e_0; e_1; e_2; e_3; e_0]$ is a horocyclic automorphism of the tree T_{e_0} . By Lemma 3.3, the projectivity u is a non-trivial permutation of ∂T_{e_0} , so that u is indeed a non-trivial automorphism of T_{e_0} . \square

4.2 \tilde{A}_2 -buildings with a linear projectivity group are Bruhat–Tits

Theorem 4.7. *Let X be a thick, locally finite, Euclidean building of type \tilde{A}_2 . Let (Π, T) be its projectivity group and $P = \overline{\Pi} \leq \text{Aut}(T)$ denote the closure of the projectivity group of X in the automorphism group of the panel-tree T , endowed with the compact-open topology.*

If P has an open subgroup admitting a continuous, faithful, finite-dimensional representation over a local field, then X is a Bruhat–Tits building.

By the classification of Bruhat–Tits buildings, one knows that X is then the Euclidean building associated with $\text{PGL}_3(D)$, where D is a finite-dimensional division algebra over a local field (see [73, Chapter 28]). Notice moreover that the converse to Theorem 4.7 holds: indeed, if X is the Euclidean building associated with $\text{PGL}_3(D)$, then its projectivity group is $\text{PSL}_2(D)$ acting on its Bruhat–Tits tree (this follows from Proposition 2.3 from [44]).

Proof of Theorem 4.7. By Proposition 3.1, the projectivity group $(\Pi, \partial T)$ is 3-transitive. In particular M is 3-transitive on ∂T . We may therefore invoke Proposition 3.8, which ensures that $(P, \partial T)$ is a Moufang set.

By Proposition 4.1, the group Π contains a non-trivial element u which is a horocyclic automorphism of T . By Proposition 3.8(iii), that element u belongs to a root group U_e of P , for some end $e \in \partial T$. Since $(P, \partial T)$ is Moufang, it follows that U_e is normal in P_e . In particular $U_e \cap \Pi_e$ is normal in Π_e . Since Π is 3-transitive on ∂T , it follows that Π_e is 2-transitive on $\partial T \setminus \{e\}$, and hence any non-trivial normal subgroup of Π_e is transitive on $\partial T \setminus \{e\}$. We have seen that $U_e \cap \Pi_e$ is a non-trivial normal subgroup of Π_e , and is thus transitive on $\partial T \setminus \{e\}$. Since the root group U_e is sharply transitive on $\partial T \setminus \{e\}$, we must have $U_e \leq \Pi_e$. It follows that $(\Pi, \partial T)$ is a Moufang set. Therefore the building at infinity Δ is Moufang by Theorem 3.2. Equivalently X is Bruhat–Tits. \square

5 Some more topological spaces associated with \tilde{A}_2 -buildings

Throughout this section, we let X be a locally finite \tilde{A}_2 -building and $\Gamma \leq \text{Aut}(X)$ be a discrete subgroup acting cocompactly on X . The goal of this section is to define auxiliary topological spaces associated with the pair (X, Γ) and needed for subsequent developments. In particular we define in §5.1 the space \mathcal{F} consisting of *marked flats* and in §5.2 the space \mathcal{S} consisting of *marked wall trees*. It turns out that the space \mathcal{S} is too big for our purposes. We therefore define the notion of *restricted marked wall trees* and study the space of such objects in §5.3. Unfortunately, this makes our discussion a bit technical. With the hope to help the reader to overcome those technicalities, we provide in §5.4 a broad overview of the spaces we consider and the maps connecting them.

5.1 The space of marked flats

We fix a model flat Σ for X , a base vertex $0 \in \Sigma$ and a sector Q^+ based at 0. The direction of Q^+ is called the **positive Weyl chamber**. The sector of Σ based at 0 and opposite Q^+ is denoted by Q^- . We let A be the group of simplicial translations of Σ (not necessarily type-preserving). The group A acts sharply transitively on the vertices of Σ . We let W be the group of type-preserving automorphisms of Σ fixing 0. It is called the **(spherical) Weyl group** of X and is isomorphic to the symmetric group S_3 .

Definition 5.1. A **marked flat** in X is a simplicial (not necessarily type-preserving) embedding $\Sigma \rightarrow X$. We denote by \mathcal{F} the set of marked flats in X and endow it with the topology of pointwise convergence.

Remark 5.2. By Theorem 2.3 the image of a marked flat in X is indeed a flat in X .

Note that \mathcal{F} is locally compact as X is locally finite. In fact it is easy to see how \mathcal{F} is decomposed into a disjoint union of compact open subspaces. We define a map

$$\mathcal{F} \rightarrow X^{(0)} : \phi \mapsto \phi(0).$$

For each $x \in X^{(0)}$ we let \mathcal{F}_x denote the preimage of x under this map. Clearly the subsets \mathcal{F}_x are compact open and $\mathcal{F} = \sqcup \mathcal{F}_x$.

The group $W \ltimes A$ acts on \mathcal{F} by precomposition and Γ acts on \mathcal{F} by postcomposition. Thus these two actions commute. We endow \mathcal{F}/Γ and \mathcal{F}/A with the quotient topologies. Note that \mathcal{F}/Γ is a $W \ltimes A$ -space and \mathcal{F}/A is a $\Gamma \times W$ -space.

Lemma 5.3. *The space $\Gamma \backslash \mathcal{F}$ is a compact $W \ltimes A$ -space.*

Proof. The union of the spaces \mathcal{F}_x taken over a finite set of representatives of the Γ -orbits in $X^{(0)}$ is compact, and maps surjectively onto $\Gamma \backslash \mathcal{F}$. The latter is thus indeed compact. \square

Recall that two chambers in Δ are called **opposite** if they are at maximal distance in the graph Δ (i.e. at distance 3). We denote by $\Delta^{2,\text{op}} \subset \text{Ch}(\Delta) \times \text{Ch}(\Delta)$ the set of pairs of opposite chambers, and endow it with the topology induced by the product topology. This is a locally compact topology. We consider the map

$$\theta : \mathcal{F} \rightarrow \Delta^{2,\text{op}} : \phi \mapsto ([\phi(Q^+)], [\phi(Q^-)]),$$

where $[\phi(Q^+)]$ denotes the equivalence class of the sector $\phi(Q^+) \subset X$ which is a chamber in $\text{Ch}(\Delta)$ and similarly for $[\phi(Q^-)]$ which is an opposite chamber.

Lemma 5.4. *The map $\theta : \mathcal{F} \rightarrow \Delta^{2,\text{op}}$ induces a $\Gamma \times W$ -equivariant homeomorphism between \mathcal{F}/A and $\Delta^{2,\text{op}}$.*

Proof. We first check that θ is continuous. Assume that $\phi_n \in \mathcal{F}$ are embeddings of Σ converging to ϕ . Then there exists x such that for n large enough we have $\phi_n(0) = x$, and for even larger n , ϕ_n is constant on the ball of radius r around x . In particular $\phi_n(Q^+)$ and $\phi(Q^+)$ are in $\Omega_x(y)$ for some y at distance r from x . Hence $\phi_n(Q^+)$ converges to $\phi(Q^+)$, and similarly for $\phi_n(Q^-)$.

By Lemma 2.7 θ is surjective and its fibers are exactly the A -orbits on \mathcal{F} . By the definition of the quotient topology we thus obtain a continuous bijection $\theta' : \mathcal{F}/A \rightarrow \Delta^{2,\text{op}}$. We need to show that θ' is an open map. For $x \in X^{(0)}$ we denote by \mathcal{F}'_x the image of \mathcal{F}_x in \mathcal{F}/A . Note that $\{\mathcal{F}'_x\}$ is an open cover of \mathcal{F}/A . Thus it is enough to show that $\theta'|_{\mathcal{F}'_x}$ is open. By compactness, $\theta'|_{\mathcal{F}'_x}$ is a homeomorphism of \mathcal{F}'_x onto its image, thus it is enough to show that $\theta'|_{\mathcal{F}'_x}$ has an open image. Note that $\theta'(\mathcal{F}'_x) = \theta(\mathcal{F}_x)$ is the union of all $(\Omega_x(y) \times \Omega_x(y')) \cap \Delta^{2,\text{op}}$, where y and y' are points in an apartment containing x which are symmetric relatively to x in that apartment. Hence this set is indeed open in $\Delta^{2,\text{op}}$. \square

5.2 The space of marked wall trees

There are two types of vertices in the spherical building Δ , which we called *points* and *lines* in Definition 2.2. From now on, we adopt the convention to denote these two types by $+$ and $-$ respectively. We let Δ_+ (resp. Δ_-) be the set of vertices of type $+$ (resp. of type $-$) in Δ . The natural maps $\text{Ch}(\Delta) \rightarrow \Delta_+$ and $\text{Ch}(\Delta) \rightarrow \Delta_-$, associating to each a chamber its two boundary vertices, are $\text{Aut}(X)$ -equivariant, continuous and surjective (see [23, Prop. 3.5]). We denote by $\Omega_x^+(y)$ and $\Omega_x^-(y)$ the image in Δ_+ and Δ_- of the basic open sets $\Omega_x(y)$.

Fix $u \in \Delta_+$. Choose an apartment F in X whose boundary contains u , and a line ℓ in F pointing at u . Then consider the union of all lines in X which are parallel (i.e. at constance distance) to ℓ . This set is isometric to $T_\ell \times \mathbf{R}$, where T_ℓ is the wall tree of ℓ , and is thus isometric to $T \times \mathbf{R}$ (see Definition 3.4). Conversely, using Theorem 2.3 it is not hard to see that any isometric embedding of $T \times \mathbf{R}$ in X is obtained in a similar way. We regard \mathbf{R} as a simplicial complex in the usual way (the integer points being the vertices) and regard $T \times \mathbf{R}$ as a simplicial complex accordingly.

Definition 5.5. A *marked wall tree* in X is a simplicial embedding $\phi: T \times \mathbf{R} \rightarrow X$ such that the limit of $\phi(t, s)$ when s tends to $+\infty$ is a vertex of type $+$. We denote by $\widetilde{\mathcal{S}}$ the set of marked wall trees in X and endow it with the topology of pointwise convergence.

Note that $\widetilde{\mathcal{S}}$ is locally compact as X is locally finite. In fact it is easy to see how it decomposes into a disjoint union of compact open subspaces. We fix a base point $o \in T$ and define a map

$$\widetilde{\mathcal{S}} \rightarrow X^{(0)} : \phi \mapsto \phi(o, 0). \quad (1)$$

For each $x \in X^{(0)}$ we let $\widetilde{\mathcal{S}}_x$ denote the preimage of x under this map. Clearly the subsets $\widetilde{\mathcal{S}}_x$ are compact open and $\widetilde{\mathcal{S}} = \sqcup \widetilde{\mathcal{S}}_x$.

The group $(\mathbf{Z}, +)$ acts naturally on $T \times \mathbf{R}$ via its action on \mathbf{R} . We denote its image in the isometry group of $T \times \mathbf{R}$ by S . We get an action of $\text{Aut}(T) \times S$ on $\widetilde{\mathcal{S}}$ by precompositions which commutes with the Γ action by postcompositions. We endow $\widetilde{\mathcal{S}}/\Gamma$ and $\widetilde{\mathcal{S}}/(\text{Aut}(T) \times S)$ with the quotient topologies. Note that $\widetilde{\mathcal{S}}/\Gamma$ is a $\text{Aut}(T) \times S$ -space and $\widetilde{\mathcal{S}}/A$ is a Γ -space.

Lemma 5.6. *The space $\widetilde{\mathcal{S}}/\Gamma$ is a compact $\text{Aut}(T) \times S$ -space.*

Proof. This follows at once from the fact that Γ has a finite fundamental domain in $X^{(0)}$ and the compactness of the spaces $\widetilde{\mathcal{S}}_x$. \square

Let Δ_\pm^{op} be the set of pairs of opposite vertices (u, v) with $u \in \Delta_+$ and $v \in \Delta_-$. This is an open subset of the compact space $\Delta_+ \times \Delta_-$, hence a locally compact space. We define a map

$$\sigma: \widetilde{\mathcal{S}} \rightarrow \Delta_\pm^{\text{op}} : \phi \mapsto (\phi^+, \phi^-) = \left(\lim_{s \rightarrow +\infty} \phi(o, s), \lim_{s \rightarrow -\infty} \phi(o, s) \right), \quad (2)$$

where $\lim_{s \rightarrow +\infty} \phi(o, s)$ represents the vertex at infinity of the ray $(\phi(o, s))_{s \geq 0}$ and similarly for $\lim_{s \rightarrow -\infty} \phi(o, s) \in \Delta_-$.

Lemma 5.7. *The map $\widetilde{\mathcal{S}} \rightarrow \Delta_\pm^{\text{op}}$ induces a Γ -equivariant homeomorphism*

$$\widetilde{\mathcal{S}}/(\text{Aut}(T) \times S) \xrightarrow{\sim} \Delta_\pm^{\text{op}}.$$

Proof. The fact that this map is bijective follows easily from the fact that there exists a wall tree between any two opposite vertices, and that $\text{Aut}(T) \times S$ acts transitively on the set of marked wall trees with a fixed image. Let us check that the map σ is continuous. Let ϕ_n be elements of \mathcal{S} , with $\phi_n \rightarrow \phi$. Let $u_n = \lim_{s \rightarrow +\infty} \phi_n(o, s)$ and $u = \lim_{s \rightarrow +\infty} \phi(o, s)$. Then for every $r > 0$, for n large enough, ϕ_n is constant on a ball of radius r around $(o, 0)$. If y is a point in this ball, it follows that $u_n \in \Omega_x^+(y)$ for n large enough. Similarly, $u \in \Omega_x^+(y)$, so that (u_n) converges to u .

Next, we have to prove that the map $\sigma': \widetilde{\mathcal{S}}/(\text{Aut}(T) \times S) \xrightarrow{\sim} \Delta_{\pm}^{\text{op}}$ is open. As in the proof of Lemma 5.4, it suffices by compactness to prove that $\sigma(\widetilde{\mathcal{S}}_x)$ is open, for every $x \in X^{(0)}$. Indeed, $\sigma(\widetilde{\mathcal{S}}_x)$ is a union of all $\Omega_x^+(y) \times \Omega_x^-(y') \cap \Delta_{\pm}^{\text{op}}$, where y and y' are vertices such that x belongs to the geodesic segment $[y, y']$, hence open. \square

Definition 5.8. We let $\overline{\Delta}_+$ be the set of pairs (v, f) where $v \in \Delta_+$ and f is a simplicial isomorphism from T to the panel tree T_v . We let π_+ be the map $\widetilde{\mathcal{S}} \rightarrow \overline{\Delta}_+$ given by $\pi_+(\phi) = (v, f)$ where v is the common limit of $\phi(x, t)$ when t tends to $+\infty$, and $f(x)$ is the class of the geodesic line $\phi(x, \cdot)$ in T_v . We define similarly $\overline{\Delta}_-$ and π_- . We endow $\overline{\Delta}_+$ and $\overline{\Delta}_-$ with the quotient topologies obtained under π_+ and π_- .

Lemma 5.9. Fix $u \in \Delta_+$. Let F_u be the set of simplicial isomorphisms from T to T_u , endowed with the topology of pointwise convergence. Then the map $F_u \rightarrow \overline{\Delta}_+$ defined by $f \mapsto (u, f)$ is a homeomorphism on its image.

Proof. Let $f_n \in F_u$ be such that (f_n) converges to f . Fix a vertex $v \in \Delta_-$ opposite to u . Let $\phi_n \in \mathcal{S}$ be such that for every n , $\lim_{s \rightarrow -\infty} \phi_n(x, s) = v$ and $\pi_+(\phi_n) = (u, f_n)$. Let $x \in T$. Then $\phi_n(x, \cdot)$ is a parametrization of a geodesic from u to v which, for n large enough, corresponds to $f(x)$. So up to translating ϕ_n we may and shall assume that $\phi_n(x, 0)$ is eventually constant. Then ϕ_n converges to some $\phi \in \widetilde{\mathcal{S}}$ with $\pi_+(\phi) = (u, f)$.

Conversely, assume that ϕ_n converges to ϕ with $\pi_+(\phi_n) = (u, f_n)$ and $\pi_+(\phi) = (u, f)$. Let $x \in T$. Then we have for n large enough $\phi_n(x, 0) = \phi(x, 0)$. Since $f_n(x)$ (resp. $f(x)$) is the projection of $\phi_n(x, 0)$ (resp. $\phi(x, 0)$) to T_u , we get immediately that f_n converges to f . \square

It will be convenient for us to consider various fiber products. If A, B, C are topological spaces, with continuous maps $p_A : A \rightarrow C$ and $p_B : B \rightarrow C$, the fiber product $A \times_C B$ is defined as the set $\{(a, b) \in A \times B \mid p_A(a) = p_B(b)\}$. It is equipped with the topology induced from the product topology.

Lemma 5.10. The map $\widetilde{\mathcal{S}}/S \rightarrow \Delta_{\pm}^{\text{op}} \times_{\Delta_+} \overline{\Delta}_+$ which associates to ϕ the pair $((\phi^+, \phi^-), \pi_+(\phi))$ is a homeomorphism.

Similarly, the map $\widetilde{\mathcal{S}}/S \rightarrow \Delta_{\pm}^{\text{op}} \times_{\Delta_-} \overline{\Delta}_-$ which associates to ϕ the pair $((\phi^+, \phi^-), \pi_-(\phi))$ is a homeomorphism.

Proof. We argue for $\Delta_{\pm}^{\text{op}} \times_{\Delta_+} \overline{\Delta}_+$ only, as the other argument is similar. The map of the Lemma is bijective: every pair of opposite vertices is linked by geodesic forming a tree, and given a pair of opposite vertices, there is an element of $\widetilde{\mathcal{S}}$ realizing a given embedding of the tree, and that element is unique modulo S . Since both maps $\widetilde{\mathcal{S}}/S \rightarrow \Delta_{\pm}^{\text{op}}$ and $\widetilde{\mathcal{S}}/S \rightarrow \overline{\Delta}_+$ are continuous, their product also is continuous.

We now have to check that this map is open. Again, by compactness, it suffices to prove that the image of $\widetilde{\mathcal{S}}_x$ in $\Delta_{\pm}^{\text{op}} \times_{\Delta_+} \overline{\Delta}_+$ is an open set. But this set is exactly the set of pairs whose first coordinate is in $\sigma(\widetilde{\mathcal{S}}_x)$. Hence it is open. \square

Lemma 5.11. *The map $\Delta_{\pm}^{\text{op}} \times_{\Delta_+} \overline{\Delta}_+ \rightarrow \overline{\Delta}_-$, which associates to an element (u, v, f) with $f: T \rightarrow T_v$ the pair (v, f') where $f': T \rightarrow T_u$ is the map obtained by composing f with the identification $T_v \rightarrow T_u$, is continuous.*

Proof. The space $\Delta_{\pm}^{\text{op}} \times_{\Delta_+} \overline{\Delta}_+$ is homeomorphic to $\widetilde{\mathcal{S}}/S$ by Lemma 5.10. The map of the Lemma is transported by this homeomorphism to the map $\pi_- : \widetilde{\mathcal{S}}/S \rightarrow \overline{\Delta}_-$, which is continuous. \square

Proposition 5.12. *Let $V_n(u, v)$ be the set of sequences (u_1, u_2, \dots, u_n) of vertices which are consecutively opposite, and such that u_1 is opposite u and u_n is opposite v . The set $V_n(u, v)$ is equipped with the topology induced from the product topology. Let also $M(u, v)$ be the set of simplicial isomorphisms of T_u to T_v , with the topology of pointwise convergence.*

Then the map

$$V_n(u, v) \rightarrow M(u, v) : (u_1, u_2, \dots, u_n) \mapsto [u; u_1; u_2; \dots; u_n; v]$$

is continuous.

Proof. Without loss of generality, we may and shall assume that u and v are of type $+$ (so that n is odd). Let V_n be the set of sequence of n consecutively opposite vertices, starting from a vertex of type $+$. Note that V_n is naturally identified to the fiber product $\Delta_{\pm}^{\text{op}} \times_{\Delta_-} \Delta_{\pm}^{\text{op}} \times_{\Delta_+} \dots \times_{\Delta_-} \Delta_{\pm}^{\text{op}}$ (with n factors).

Let $P_n : V_n \times_{\Delta_+} \overline{\Delta}_+ \rightarrow \overline{\Delta}_+$ be the map which associates to a sequence $(u_1, \dots, u_n) \in V_n$ and $(u_n, f) \in \overline{\Delta}_+$ the map $f' : T \rightarrow T_{u_1}$ obtained by composing the map f with the perspectivity $[u_n; u_{n-1}; \dots; u_1]$. Applying Lemma 5.11 (and its symmetric statement with $+$ and $-$ exchanged), and using the associativity of the fiber product, we see that the map P_n is continuous.

Let F_u (resp. F_v) be the set of simplicial maps $T \rightarrow T_u$ (resp. $T \rightarrow T_v$). Let $F(u, v)$ be the set of continuous maps from F_u to F_v . The sets F_u and F_v are equipped with the topology of pointwise convergence and the set $F(u, v)$ with the compact open topology. Let $M(u, v)$ be the set of maps $T_u \rightarrow T_v$, with the topology of pointwise convergence. There is a map $M(u, v) \rightarrow F(u, v)$, sending m to the map $f \mapsto m \circ f$. This map is a homeomorphism on its image.

Specializing to the case when the first vertex is u and the last one is v , we get a new map $P'_n : V_n(u, v) \rightarrow F(u, v)$ as follows: $P'_n(u_1, \dots, u_n)$ sends $f : T \rightarrow T_u$ to the map $f' : T \rightarrow T_v$ defined by $f' = [u; u_1; \dots; u_n; v] \circ f$. Following the definitions, we see that $P_{n+2}(v, u_n, \dots, u_1, u, f) = (v, P'_n(u_1, \dots, u_n)(f))$. Thus P'_n is continuous. The Proposition follows. \square

5.3 The space of restricted marked wall trees

In the next section we will construct measures on the various spaces we consider here. There is a natural measure on $\widetilde{\mathcal{S}}$ that we could construct. However, we will not do it, as this measure fails to satisfy the ergodic properties that we desire. Instead we will consider a certain subspace of $\widetilde{\mathcal{S}}$, to be denoted \mathcal{S} , on which the analogous measure will be better behaved. This subsection is devoted to the definition and the study of this subspace and its properties.

For the next definition, note that $\text{Aut}(T) \times S$ acts on $\widetilde{\mathcal{S}}$ by precomposition and Γ acts on $\widetilde{\mathcal{S}}$ by postcomposition.

Definition 5.13. Two embeddings $\phi, \psi \in \widetilde{\mathcal{S}}$ are **positively equivalent** if for every large enough $t \in \mathbf{R}$, for every $x \in T$, we have $\phi(x, t) = \psi(x, t)$. The embeddings ϕ and ψ are **negatively equivalent** if for every large enough $t \in \mathbf{R}$, for every $x \in T$, we have $\phi(x, -t) = \psi(x, -t)$. We denote by \sim_+ the positive equivalence relation and by \sim_- the negative equivalence relation. We denote the equivalence relation on $\widetilde{\mathcal{S}}$ generated by \sim_+, \sim_- and the action of $S \times \Gamma$ by \sim .

Lemma 5.14. *The equivalence relation \sim is $\text{Aut}(T) \times S \times \Gamma$ -invariant. Moreover, every equivalence class is $S \times \Gamma$ -invariant, and $\text{Aut}(T)$ acts transitively on $\widetilde{\mathcal{S}}/\sim$.*

Proof. By construction, every \sim -class is $S \times \Gamma$ -invariant. Clearly \sim_+ and \sim_- are $\text{Aut}(T)$ invariant. Moreover the $S \times \Gamma$ -equivalence relation is also invariant, since the $S \times \Gamma$ -action and the $\text{Aut}(T)$ -action commute. It follows that \sim is also $\text{Aut}(T)$ -invariant.

It remains to show that $\text{Aut}(T)$ is transitive on $\widetilde{\mathcal{S}}/\sim$. Let thus $\phi_1, \phi_2 \in \widetilde{\mathcal{S}}$. We must find $\alpha \in \text{Aut}(T)$ such that $\phi_1 \circ \alpha \sim \phi_2$. Set $(\phi_i^+, \phi_i^-) = (u_i, v_i) \in \Delta_{\pm}^{\text{op}}$. We choose $u \in \Delta_-$ which is opposite to both v_1, v_2 and $v \in \Delta_+$ which is opposite to both u, u_2 (see Lemma 2.8). We next choose successively $\psi_1, \psi_2, \psi_3, \psi_4 \in \widetilde{\mathcal{S}}$ with:

- $\psi_1^+ = u$ and $\psi_1 \sim_- \phi_1$;
- $\psi_2^- = v$ and $\psi_2 \sim_+ \psi_1$;
- $\psi_3^+ = u_2$ and $\psi_3 \sim_- \psi_2$;
- $\psi_4^- = v_2$ and $\psi_4 \sim_+ \psi_3$.

Thus we have $\phi_1 \sim_- \psi_1 \sim_+ \psi_2 \sim_- \psi_3 \sim_+ \psi_4$, so that $\phi_1 \sim \psi_4$. Moreover $(\psi_4^+, \psi_4^-) = (u_2, v_2) = (\phi_2^+, \phi_2^-)$. The desired conclusion follows since $\text{Aut}(T) \times S$ is transitive on each fiber of the map $\widetilde{\mathcal{S}} \rightarrow \Delta_{\pm}^{\text{op}}$ by Lemma 5.7. \square

Fix $\gamma \in \Gamma$ and $u \in \Delta_-$. Then γ induces a map $T_u \rightarrow T_{\gamma u}$. Let $(\gamma u, v_1, \dots, v_n)$ be a sequence of consecutively opposite vertices. We denote by $[\gamma : u; v_1; \dots; v_n]$ the map $T_u \rightarrow T_{v_n}$ obtained by first applying γ and then the perspectivity $[\gamma u; v_1; \dots; v_n]$.

Lemma 5.15. *Fix $\phi, \phi' \in \widetilde{\mathcal{S}}$ satisfying $\phi \sim \phi'$ and denote $\pi_-(\phi) = (u, f)$, $\pi_-(\phi') = (u', f')$. Then there exist $\gamma \in \Gamma$ and v_1, \dots, v_n such that $(\gamma\phi_-, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n), (v_n, \phi'_-)$ are all opposite pairs and $f' = [\gamma : u; v_1; \dots; v_n; u] \circ f$.*

Proof. By the definition of \sim there exists a sequence $(\phi = \phi_0, \phi_1, \dots, \phi_{n-1}, \phi_n = \phi')$ such that ϕ_i and ϕ_{i+1} are equivalent for \sim_+ or \sim_- or in the same $S \times \Gamma$ -orbit. Since \sim_- and \sim_+ are preserved by $S \times \Gamma$ (see Lemma 5.14) we may assume that $\phi_1 = (s, \gamma)\phi$ and for $i > 1$, ϕ_i and ϕ_{i+1} are either \sim_+ or \sim_- equivalent. Since \sim_+ and \sim_- are transitive relations we can assume further that they alternate. The lemma now follows. \square

Lemma 5.16. *Fix an \sim -equivalence class \mathcal{S}' and let $M' < \text{Aut}(T)$ be the stabilizer of \mathcal{S}' . Fix $\phi \in \mathcal{S}'$ and set $\pi_-(\phi) = (u, f) \in \bar{\Delta}_-$. Consider the group $M'_\phi = fM'f^{-1} < \text{Aut}(T_u)$. Then*

$$M'_\phi = \{[\gamma : u; v_1; \dots; v_n; u] \mid (\gamma u, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n), (v_n, u) \text{ are all opposite pairs}\}.$$

In particular, the projectivity group $\Pi(u)$ is contained in M'_ϕ .

Proof. Fix $m \in M'$. We have $\phi \circ m \sim \phi$ by the definition of M' . Moreover for any $m' \in \text{Aut}(T_u)$, the equality $m' \circ f = \pi_-(\phi \circ m)$ implies $m' = f \circ m \circ f^{-1}$. We thus need to construct for every $\phi' \sim \phi$ an element of the form $[\gamma : u; v_1; \dots; v_n; u]$ such that $\phi' = [\gamma : u; v_1; \dots; v_n; u] \circ \phi$. This indeed follows from Lemma 5.15. \square

We wish to consider a single equivalence class in $\widetilde{\mathcal{S}}$. A technical problem is that such an equivalence class will not be closed. The obvious solution is to replace the equivalence relation \sim by its closure (as a subset of $\widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$). Note that in general the closure of an equivalence relation need not be an equivalence relation. However the situation is better when a topological group is acting continuously on the ambient space and transitively on the quotient space.

Definition 5.17. We let \simeq be the closure of \sim as a subset of $\widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$.

Lemma 5.18. *The relation \simeq is an equivalence relation and each \simeq -equivalence class is closed in $\widetilde{\mathcal{S}}$. In fact, the \simeq -equivalence classes are the closures of the \sim -equivalence classes. In particular, every \simeq -equivalence class is $S \times \Gamma$ -invariant.*

Proof. Recall from Lemma 5.14 that $\text{Aut}(T)$ acts transitively on $\widetilde{\mathcal{S}}/\sim$. Therefore, the orbit map induces an equivariant bijection $\widetilde{\mathcal{S}}/\sim \rightarrow \text{Aut}(T)/M'$, where M' is the subgroup of $\text{Aut}(T)$ stabilizing some \sim -equivalence class. Let then M be the closure of M' in $\text{Aut}(T)$. We have canonical maps

$$\widetilde{\mathcal{S}} \rightarrow \widetilde{\mathcal{S}}/\sim \rightarrow \text{Aut}(T)/M' \rightarrow \text{Aut}(T)/M.$$

We may now identify \simeq with the fiber equivalence relation of the map $\widetilde{\mathcal{S}} \rightarrow \text{Aut}(T)/M$, and the required assertions follows. \square

The following crucial definition involves a choice.

Definition 5.19. We fix once and for all an \simeq -equivalence class in $\widetilde{\mathcal{S}}$ and denote it \mathcal{S} . We call the elements of \mathcal{S} the **restricted marked wall trees**. We let $M < \text{Aut}(T)$ be the stabilizer of \mathcal{S} .

Lemma 5.20. *The space \mathcal{S} is closed in $\widetilde{\mathcal{S}}$ and invariant under the action of $M \times S \times \Gamma$. The group M is the closure of the group M' described in Lemma 5.16. The restriction of the map $\widetilde{\mathcal{S}} \rightarrow \Delta_{\pm}^{\text{op}}$ to \mathcal{S} gives a Γ -equivariant homeomorphism $\mathcal{S}/(M \times S) \xrightarrow{\sim} \Delta_{\pm}^{\text{op}}$.*

Proof. The embedding $\mathcal{S} \rightarrow \widetilde{\mathcal{S}}$ descends to a continuous map $\mathcal{S}/M \rightarrow \widetilde{\mathcal{S}}/\text{Aut}(T)$ which is injective by the definition of M . That map is moreover surjective in view of 5.14. Therefore we have a Γ -equivariant homeomorphism $\mathcal{S}/(M \times S) \rightarrow \widetilde{\mathcal{S}}/(\text{Aut}(T) \times S)$. The last required assertion now follows from Lemma 5.7. \square

Theorem 5.21. *Γ acts properly on \mathcal{S} and \mathcal{S}/Γ is a compact $M \times S$ -space. The group M is a unimodular group which acts 3-transitively on the boundary of the tree T . In particular M contains a closed, cocompact, normal subgroup M^+ which is compactly generated, topologically simple and non-discrete. Further more, if X is not Bruhat–Tits then M^+ has no non-trivial continuous linear representation over any local field.*

Proof. The first sentence follows by Lemma 5.20 and Lemma 5.6. By Lemma 5.20 the group M is the closure of the group M' and by Lemma 5.16 the group M contains a copy Π of the projectivity group $\Pi(u)$ for some $u \in \Delta_-$. It follows by Lemma 3.5 that M acts 3-transitively on ∂T . The group M^+ is described in Theorem 3.7 where its various properties are described, as well as the unimodularity of M . We are left to show that the non-linearity of M^+ in case X is not Bruhat–Tits. To this end we consider the topologically simple group $\bar{\Pi}^+$ which is obtained by applying Theorem 3.7 to $\bar{\Pi}$. As M/M^+ is profinite and $\bar{\Pi}^+$ topologically simple we get that $\bar{\Pi}^+ < M^+$. By Theorem 4.7 $\bar{\Pi}^+$ has no linear representations over local fields. We deduce that M^+ shares this property, as it is topologically simple. \square

Definition 5.22. We call the $M \times S$ -space \mathcal{S}/Γ the **singular Cartan flow**.

5.4 The broad picture

In this subsection we pack together the various Γ -spaces constructed previously in the following commutative diagram of Γ -equivariant maps:

$$\begin{array}{ccccccc}
 X^{(0)}/\Gamma & \longleftarrow & \mathcal{F}/\Gamma & \longleftarrow & \mathcal{S}/\Gamma & & \\
 \uparrow & & \uparrow & & \uparrow & & \\
 X^{(0)} & \longleftarrow & \mathcal{F} & \longleftarrow & \mathcal{S} & & \\
 & & \downarrow & & \downarrow & \searrow & \\
 & & & & \mathcal{S}/S & \longrightarrow & \bar{\Delta}_+ & \longrightarrow & \bar{\Delta}_- \\
 & & \swarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Ch}(\Delta) & \xleftarrow{p_1, p_2} & \Delta^{2, \text{op}} & \longrightarrow & \Delta_{\pm}^{\text{op}} & \longrightarrow & \Delta_+ & \longrightarrow & \Delta_-
 \end{array} \tag{3}$$

In order to keep our notation as light as possible we will not name most of the arrows in Diagram (3), but in what follows we will make clear what they are.

In the beginning of §5.1 we have fixed the model apartment Σ and a positive Weyl chamber Q^+ emanating from $0 \in \Sigma$. We defined $\mathcal{F} \rightarrow X^{(0)}$ by

$$\mathcal{F} = \{\text{simplicial embeddings } \Sigma \rightarrow X\} \ni \phi \mapsto \phi(0) \in X^{(0)}.$$

We further defined the map $\mathcal{F} \rightarrow \Delta^{2, \text{op}}$ by

$$\mathcal{F} = \{\text{simplicial embeddings } \Sigma \rightarrow X\} \ni \phi \mapsto ([\phi(Q^+)], [\phi(Q^-)]) \in \Delta^{2, \text{op}},$$

where $[\phi(Q^+)]$ denotes the equivalence class of the section $\phi(Q^+) \subset X$ which is a point in Δ , similarly for $[\phi(Q^-)]$ which is an opposite point.

In the beginning of §5.2 we fixed the model simplicial complex $T \times \mathbf{R}$ and a base point $o \in T$. We now relate Σ and $T \times \mathbf{R}$. We denote by ℓ the unique wall through 0 in Σ such that $\ell \cap \overline{Q^+} = \{0\}$ and fix a simplicial embedding $\Sigma \hookrightarrow T \times \mathbf{R}$ such that the image of ℓ is $\{o\} \times \mathbf{R} \subset T \times \mathbf{R}$. Recall that $\mathcal{S} \subset \widetilde{\mathcal{F}} = \{\text{simplicial embeddings } T \times \mathbf{R} \rightarrow X\}$. We define the map

$$\mathcal{S} \rightarrow \mathcal{F} : \psi \mapsto \psi|_{\Sigma}.$$

Note that the composition $\mathcal{S} \rightarrow \mathcal{F} \rightarrow X^{(0)}$ agrees with the restriction to \mathcal{S} of the map (1) given in §5.2. The maps $\mathcal{S} \rightarrow \mathcal{F} \rightarrow X^{(0)}$ are clearly Γ -equivariant (Γ acts on the source space X) and induce corresponding maps $\mathcal{S}/\Gamma \rightarrow \mathcal{F}/\Gamma \rightarrow X^{(0)}/\Gamma$. The commutativity of the upper part of Diagram (3) is obvious.

Note that the composed map $\mathcal{S} \rightarrow \mathcal{F} \rightarrow \Delta^{2,\text{op}}$ is S -invariant, thus factors through \mathcal{S}/S . This explains the map $\mathcal{S}/S \rightarrow \Delta^{2,\text{op}}$ appearing in diagram (3). We define the map $\Delta^{2,\text{op}} \rightarrow \Delta_{\pm}^{\text{op}}$ by associating with a pair of chambers $(C_1, C_2) \in \Delta^{2,\text{op}}$ the unique pair of elements $\delta_+ \in \Delta_+$ and $\delta_- \in \Delta_-$ such that δ_+ and δ_- are in the boundary of the unique flat that contains both C_+ and C_- but disjoint from their closures. By composing $\mathcal{S}/S \rightarrow \Delta^{2,\text{op}} \rightarrow \Delta_{\pm}^{\text{op}}$ we also obtain the map $\mathcal{S}/S \rightarrow \Delta_{\pm}^{\text{op}}$. Note that the composed map $\mathcal{S} \rightarrow \mathcal{S}/S \rightarrow \Delta_{\pm}^{\text{op}}$ is the restriction to \mathcal{S} of the map (2) defined in §5.2, namely

$$\phi \mapsto (\phi^+, \phi^-) = \left(\lim_{s \rightarrow +\infty} \phi(o, s), \lim_{s \rightarrow -\infty} \phi(o, s) \right) \in \Delta_{\pm}^{\text{op}}.$$

Similarly we can restrict to \mathcal{S} the maps $\widetilde{\mathcal{S}} \rightarrow \overline{\Delta}_+, \overline{\Delta}_-$ appearing in Definition 5.8 and obtain maps $\mathcal{S} \rightarrow \overline{\Delta}_+, \overline{\Delta}_-$. The latter, being S -invariant, factor through \mathcal{S}/S and we obtain the maps $\mathcal{S}/S \rightarrow \overline{\Delta}_+, \overline{\Delta}_-$ appearing in the diagram. The maps $\overline{\Delta}_+ \rightarrow \Delta_+$ and $\overline{\Delta}_- \rightarrow \Delta_-$ are the obvious ones and the maps $\text{Ch}(\Delta) \rightarrow \Delta_+$ and $\text{Ch}(\Delta) \rightarrow \Delta_-$ are the ones described in the beginning of §5.2. The commutativity of the diagram is clear from the definitions. Finally, the maps $p_1, p_2: \Delta^{2,\text{op}} \rightarrow \text{Ch}(\Delta)$ are the first and second factor projections.

Note that all the maps in diagram (3) are Γ -equivariant. Further it was observed in Lemma 5.4 that the map $\mathcal{F} \rightarrow \Delta^{2,\text{op}}$ is W -equivariant. We end this section by pointing an equivariance property of the composed map $\mathcal{S} \rightarrow \Delta^{2,\text{op}}$.

Proposition 5.23. *There exists a closed subgroup $N < M^+$ and a continuous surjective homomorphism $N \rightarrow \{1, \tau\}$, where τ is the flip acting on $\text{Ch}(\Delta) \times \text{Ch}(\Delta)$, such that the map $\mathcal{S} \rightarrow \Delta^{2,\text{op}}$ is equivariant with respect to this homomorphism. It also follows that the map $\mathcal{S}/S \rightarrow \Delta^{2,\text{op}}$ is equivariant with respect to $N \rightarrow \{1, \tau\}$.*

Proof. Note that, by the choice of ℓ , the longest element $w_0 \in W$ acts on Σ as a reflection along ℓ . The image of Σ under $T \times \mathbf{R} \rightarrow T$ is a geodesic in T . We let N be the stabilizer of this geodesic in M^+ . We let N_0 the normal subgroup of N which fixes the end points of this geodesic. By Theorem 5.21, the M^+ action on ∂T is 2-transitive, thus $[N : N_0] = 2$. Note that N is the stabilizer of the image of Σ in $T \times \mathbf{R}$ under the action of M^+ , thus N acts naturally on Σ . This gives a homomorphism $N \rightarrow W \ltimes A$ and the restriction map $\mathcal{S} \rightarrow \mathcal{F}$ is equivariant with respect to this homomorphism. By the above discussion, the image of N under $N \rightarrow W \ltimes A \rightarrow W$ consists of $\{1, w_0\}$ (and the kernel is N_0). Since $\mathcal{F} \rightarrow \Delta^{2,\text{op}}$ is W -equivariant we conclude that $\mathcal{S} \rightarrow \Delta^{2,\text{op}}$ is equivariant with respect to the surjective homomorphism $N \rightarrow \{1, w_0\}$. The proof is complete observing that w_0 acts on $\Delta^{2,\text{op}}$ through the flip. \square

6 Measuring the topology

In this section, we retain the notation from the previous one and explain how to construct natural measures on the various topological spaces considered there. In §6.1 we will give the construction of visual measures on Δ and we will use it in §6.2 to construct a Radon measure on $\Delta^{2,\text{op}}$. In §6.3 we will use the above to construct a Radon measure on Δ_{\pm}^{op} . Using the

measure on Δ_{\pm}^{op} we will be able to define a measure on \mathcal{S} and by this on the various spaces appearing in Diagram (3). This process will be carried out in §6.4.

6.1 The visual measures on $\text{Ch}(\Delta)$

Our goal is to define natural probability measures μ_x on $\text{Ch}(\Delta)$, indexed by $x \in X$. We follow the construction of J. Parkinson from [51].

Recall that A is the group of simplicial translations of our model apartment Σ and $Q^+ \subset \Sigma$ is the positive Weyl chamber. We denote by A^+ the subsemigroup of A formed by elements translating 0 to a point in Q^+ (including possibly the walls). This gives a partial order on A , defined by $\lambda > \mu$ if $\lambda - \mu \in A^+$.

Let α_1 and α_2 be the simple roots associated with the choice of the positive Weyl chamber. In other words, they are the \mathbf{Z} -linear forms on Σ whose kernel are walls delimiting Q^+ which are positive on Q^+ and obtain the value 1. We also denote by A^{++} the set of $\lambda \in A$ such that $\alpha_1(\lambda) > 0$ and $\alpha_2(\lambda) > 0$.

Let t_1 and t_2 be the two generators of A such that every $t \in A$ is written $t = \alpha_1(t)t_1 + \alpha_2(t)t_2$. We denote $\ell(t) = \alpha_1(t) + \alpha_2(t)$.

For $\lambda \in A^+$ and x a vertex in X we denote by $V_\lambda(x)$ the set of vertices $y \in X$ such that, in an apartment $F \subset X$ containing x and y , we have $\lambda(x) = y$.

There is a natural action of the Weyl group W , isomorphic to the symmetric group S_3 , on Σ . The group W is generated by the two reflections s_1 and s_2 , which act on Σ by reflections with respect to the walls corresponding to α_1 and α_2 respectively.

Lemma 6.1. *Let $\lambda \in A^{++}$ and x be a vertex in X . Then the cardinality N_λ of $V_\lambda(x)$ does not depend on x and we have*

$$N_\lambda = Kq^{2\ell(\lambda)},$$

for some constant K depending on q .

Proof. See [27, Corollary 2.2] or [50, Theorem 5.15]. □

We will need a more precise calculation. Fix a chamber $C \in \text{Ch}(\Delta)$ and a vertex $x \in X$ contained in some apartment. Consider the collection F of all apartments in X containing x and C . Each element of F could be seen as a marked flat, by choosing its unique type preserving identification with Σ taking 0 to x and Q^+ to $Q(x, C)$. For given $\lambda \in A^+$ and $w \in W$ we consider in each apartment in F the image of the element $w \circ \lambda(0) \in \Sigma$ under the associated identification. we denote by Y_w^λ the set of all vertices of X obtained this way.

Lemma 6.2. *Denoting $(i, j) = (\alpha_1(\lambda), \alpha_2(\lambda))$, we have:*

$$|Y_e^\lambda| = 1, \quad |Y_{s_1}^\lambda| = K_1 q^j, \quad |Y_{s_2}^\lambda| = K_2 q^i, \quad |Y_{s_1 s_2}^\lambda| = K_3 q^{2i+j} \quad \text{and} \quad |Y_{s_2 s_1}^\lambda| = K_4 q^{i+2j},$$

for some constants K_1 to K_4 depending only on q . In particular, for every $w \neq w_0$, the quantity $|Y_w^\lambda|/|V_\lambda(y)|$ tends to 0 as i and j tend (simultaneously) to $+\infty$.

Proof. We argue by induction on $\ell(\lambda) = i + j$. The first step of the induction is absorbed by the constants. Let us do the calculation of $|Y_{s_1}^\lambda|$, the other ones being similar. We refer to Figure 3.

Fix $\lambda = it_1 + jt_2$. Let $\lambda' = (i+1)t_1 + jt_2$. Let $Y = Y_{s_1}^\lambda$ and $Y' = Y_{s_1}^{\lambda'}$. We consider the following map ψ : to a point z' in $V_{\lambda'}(x)$, we associate the unique point $z = \psi(z')$ in the

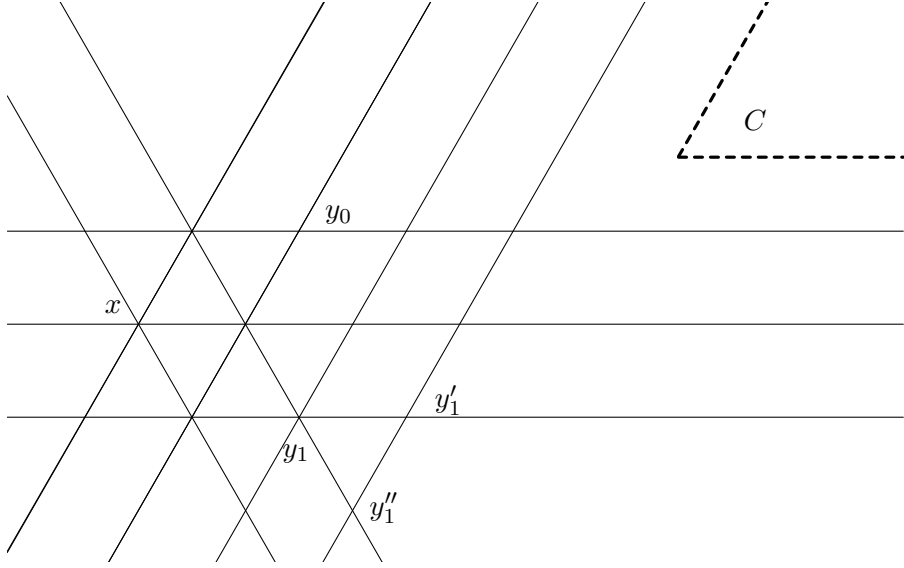


Figure 3: Counting elements in $Y_{s_1}^\lambda$

intersection of $V_\lambda(x)$ and the convex hull of x and z . The map ψ is surjective: indeed, if $z \in V_\lambda(x)$, then x and z are in some common apartment, and it is possible to find z' in this apartment such that $\psi(z') = z$. We also claim that the restriction of this map to Y' is also injective. Indeed, if $y_1 \in Y$, then any apartment containing y_1, x and C also contains the convex hull of y_1 and C , and therefore already contains a point $y'_1 \in Y'$. On the other hand, any point $y' \in Y'$ such that $\psi(y') = y_1$ is contained in some apartment containing y_1, x and C , and therefore must be equal to y'_1 . This proves the claim. Hence we have $|Y'| = |Y| = K_1 q^{2j}$.

Now let $\lambda'' = it_1 + (j+1)t_2$, and denote $Y'' = Y_{s_1}^{\lambda''}$. We still have a surjective map $Y'' \rightarrow Y$, defined in the same way as the map $Y' \rightarrow Y$. However this map is no longer injective. To count the cardinality of the fibers, choose $y_1 \in Y$. This gives a choice of $y'_1 \in Y'$, defined as $y'_1 = \psi^{-1}(y_1)$. There are q chambers which have y_1 and y'_1 as their vertices, and are not in the convex hull of C and y_1 . The third vertex of each of these chambers is a possible choice for $y''_1 \in Y''$. Hence $|Y''| = q|Y| = K_1 q^{j+1}$. \square

Definition 6.3. Let x be a vertex in X . The **visual measure** based at $x \in X$, denoted μ_x , is the unique measure on Δ such that for all $y \in V_\lambda(x)$, the set $\Omega_x(y)$ has measure $\frac{1}{N_\lambda}$.

The construction of this measure is justified more carefully in [51]; it follows from the fact that $\text{Ch}(\Delta)$ can be viewed as the projective limit of sets $V_\lambda(x)$ as λ grows in A^+ . We note also that $g \cdot \mu_x = \mu_{gx}$.

Definition 6.4. Let $C \in \text{Ch}(\Delta)$. The **horofunction** based at C , denoted h_C is a function $X \times X$ defined as follows. Let $z \in Q(x, C) \cap Q(y, C)$. Assume that $z \in V_\lambda(x) \cap V_\mu(y)$. Then $h_C(x, y) = \lambda - \mu$.

This definition does not depend on the particular choice of z (see [51, Theorem 3.4]) The horofunction $h_C(\cdot, \cdot)$ satisfies the following cocycle relation [51, Proposition 3.5]:

Lemma 6.5. For all $x, y, z \in X$ and $C \in \text{Ch}(\Delta)$, we have $h_C(x, y) = h_C(x, z) + h_C(z, y)$.

Proposition 6.6. *For x, y vertices in X , the measure μ_x and μ_y are absolutely continuous relatively to each other. Furthermore, for all $C \in \text{Ch}(\Delta)$, we have*

$$\frac{d\mu_x}{d\mu_y}(C) = q^{2\ell(h_C(x,y))}$$

Proof. See [51, Theorem 3.17]. □

Definition 6.7. We denote by μ the measure class of μ_x .

Proposition 6.8. *The set $\Delta^{2,\text{op}}$ is of conull measure (relatively to the measure class $\mu \otimes \mu$).*

Proof. For every vertex $x \in X$, let Δ_x^2 (resp. $\Delta_x^{2,\text{op}}$) be the subset of $\text{Ch}(\Delta) \times \text{Ch}(\Delta)$ composed of pairs of chambers which are in the boundary of an apartment containing x (resp. and are opposite).

Then $\text{Ch}(\Delta) \times \text{Ch}(\Delta) = \bigcup_{x \in X} \Delta_x^2$. By countability it is sufficient to prove that for every x we have $\mu_o^2(\Delta_x^{2,\text{op}}) = \mu_o^2(\Delta_x^2)$, for a fixed vertex $o \in X$. Since μ_x and μ_o are in the same measure class, it is sufficient to prove that $\mu_x^2(\Delta_x^{2,\text{op}}) = \mu_x^2(\Delta_x^2)$ for every vertex x .

By Fubini, it is enough to prove that for almost every chamber at infinity C , the set B of chambers C' such that $(C, C') \in \Delta_x^{2,\text{op}}$ is of full measure (in the set of chambers in a same apartment as C and x). Let $\mu_{x,C}$ be the measure μ_x restricted to the subset of Δ formed by chambers C' with $(C, C') \in \Delta_x^2$. We have to prove that $\mu_{x,C}(B) = \mu_{x,C}(\text{Ch}(\Delta))$.

Recall the sets Y_w^λ discussed in Lemma 6.2 and set $\Omega_x(Y_w^\lambda) = \bigcup_{y \in Y_w^\lambda} \Omega_x(y)$. It is clear that $B \subset \bigcap_{\lambda \in A^+} \Omega_x(Y_{w_0}^\lambda)$. We claim that this inclusion is actually an equation. Indeed, if $C' \in \bigcap_{\lambda \in A^+} \Omega_x(Y_{w_0}^\lambda)$, then every finite subset of $Q(x, C') \cup Q(x, C)$ is contained in some apartment, hence by Theorem 2.3 also $Q(x, C') \cup Q(x, C)$ is contained in an apartment, and then it is clear that C and C' are opposite in it.

Not that the intersection $\bigcap_{\lambda \in A^+} \Omega_x(Y_{w_0}^\lambda)$ is decreasing, in the sense that for every $\lambda, \lambda' \in A^+$ such that $\lambda' - \lambda \in A^+$, we have $\Omega_x(Y_{w_0}^{\lambda'}) \subset \Omega_x(Y_{w_0}^\lambda)$. Hence it is enough to prove that $\mu_{x,C}(\Omega_x(Y_{w_0}^\lambda))$ tends to $\mu_{x,C}(\Delta)$ when $\lambda \rightarrow \infty$. Equivalently, we need to show that for every $w \neq w_0$, the limit of $\mu_{x,C}(\Omega_x(Y_w^\lambda))$ is 0 as λ tends to infinity (simultaneously in both directions). This follows from Lemma 6.2, as $\mu_x(\Omega_x(Y_w^\lambda)) = |Y_w^\lambda|/|V_\lambda(y)|$. □

6.2 The Radon measure on $\Delta^{2,\text{op}}$

In this section we construct a natural Radon measure on \mathcal{F} , invariant by the actions of A and Γ . The existence of such a measure is well-known to experts (see for example [49] for a possible definition); its construction is nevertheless usually different than what we propose here.

Lemma 6.9. *For $(C, C') \in \Delta^{2,\text{op}}$ and x a vertex in X , the quantity*

$$\beta_x(C, C') := h_C(x, z) + h_{C'}(x, z),$$

with z a point in the apartment containing C and C' , does not depend on z .

Furthermore, we have, for every vertex $y \in X$

$$\beta_x(C, C') - \beta_y(C, C') = h_C(x, y) - h_{C'}(x, y).$$

Proof. Let us prove first that $\beta_x(C, C')$ is independent of z . Let z and z' be two points in the apartment F containing C and C' . Using the assumption that C and C' are opposite, we have that $h_C(z', z) = -h_{C'}(z', z)$.

Hence by Lemma 6.5 we have

$$\begin{aligned}\beta_x(C, C') &= h_C(x, z) + h_{C'}(x, z) \\ &= h_C(x, z') + h_C(z', z) + h_{C'}(x, z') + h_{C'}(z', z) \\ &= h_C(x, z') + h_{C'}(x, z').\end{aligned}$$

Now let us turn to the second equality. Fix $x, y \in X$. Then, using Lemma 6.5 again, we have

$$\begin{aligned}\beta_x(C, C') - \beta_y(C, C') &= h_C(x, z) + h_{C'}(x, z) - h_C(y, z) - h_{C'}(y, z) \\ &= h_C(x, z) + h_{C'}(x, z) + h_C(z, y) + h_{C'}(z, y) \\ &= h_C(x, y) + h_{C'}(z, y).\end{aligned}\quad \square$$

For a fixed vertex x in X we define the measure m_x on $\text{Ch}(\Delta) \times \text{Ch}(\Delta)$ by the formula

$$\text{d}m_x(C, C') = q^{-2\ell(\beta_x(C, C'))} \text{d}\mu_x(C) \text{d}\mu_x(C').$$

By Lemma 6.9 and Proposition 6.6, using also the fact that $\ell(t + t') = \ell(t) + \ell(t')$ for every $t, t' \in A$, we have

$$\frac{\text{d}m_x}{\text{d}m_y}(C, C') = q^{-2\ell(\beta_x(C, C')) + 2\ell(\beta_y(C, C'))} \frac{\text{d}\mu_x}{\text{d}\mu_y}(C) \frac{\text{d}\mu_x}{\text{d}\mu_y}(C') = 1.$$

Thus the measure m_x is independent on the choice of x . We get that this measure is Γ -invariant, as for all $g \in \Gamma$ we have $g\mu_x = \mu_{gx}$ and $\beta_{gx}(gC, gC') = \beta_x(C, C')$, hence $gm_x = m_{gx} = m_x$.

Using Proposition 6.8 we view m_x as a measure on $\Delta^{2, \text{op}}$ which, in view of its independence on x , we denote m . Thus m is a Γ -invariant measure on $\Delta^{2, \text{op}}$.

Recall that for a vertex $x \in X$ we defined \mathcal{F}_x to be the space simplicial embeddings $f : \Sigma \rightarrow X$ satisfying $f(0) = x$ and let \mathcal{F}'_x be the image under the projection $\mathcal{F} \rightarrow \Delta^{2, \text{op}}$. Note that $\mathcal{F} \rightarrow \Delta^{2, \text{op}}$ is injective in restriction to \mathcal{F}_x , thus induces a homeomorphism $\mathcal{F}_x \rightarrow \mathcal{F}'_x$ by Lemma 5.4.

Lemma 6.10. *We have $m(\mathcal{F}'_x) < \infty$.*

Proof. By definition, $m(\mathcal{F}'_x) = \int_{(C, C') \in \mathcal{F}'_x} q^{2\ell(\beta_x(C, C'))} \text{d}\mu_x(C) \text{d}\mu_x(C')$.

By Lemma 6.9, if $(C, C') \in \mathcal{F}'_x$, we have $\beta_x(C, C') = h_C(x, x) - h_{C'}(x, x) = 0$. Hence $m(\mathcal{F}'_x) = (\mu_x \otimes \mu_x)(\mathcal{F}'_x) < \infty$. \square

Corollary 6.11. *The measure m is a Γ -invariant Radon measure on the locally compact space $\Delta^{2, \text{op}}$.*

Proof. The only thing that does not immediately follow from the discussion above is the fact that m is finite on compact sets. This follows from Lemma 6.10, as $\{\mathcal{F}'_x \mid x \in X^{(0)}\}$ is an open cover of $\Delta^{2, \text{op}}$. \square

6.3 The Radon measure on Δ_{\pm}^{op}

The first part of this section is devoted to the construction of a Radon measure m_{\pm} on Δ_{\pm}^{op} , see Definition 6.14 and Lemma 6.15 below. The construction is similar to the construction of m given in the previous section. The second part of this section is a preparation for the proof that the measures m and m_{\pm} are compatible which will be given in the next section, see Lemma 6.21.

For $u \in \Delta_+$ (resp. $v \in \Delta_-$), we define the horofunction h_u (resp. h_v) by $h_u(x, y) = \alpha_1(h_C(x, y))$, for any chamber C adjacent to u (resp. $h_v(x, y) = \alpha_2(h_C(x, y))$ for any chamber C adjacent to v). It is easy to check that this quantity does not depend on the choice of C . Let $\text{pr}_+ : \text{Ch}(\Delta) \rightarrow \Delta_+$ be the natural projection. We define $\mu_{x,+} = (\text{pr}_+)_* \mu_x$. Similarly, we define $\mu_{x,-} = (\text{pr}_-)_*(\mu_x)$, where $\text{pr}_- : \text{Ch}(\Delta) \rightarrow \Delta_-$. Recall also the notation $\Omega_x^+(y) = \text{pr}_+(\Omega_x(y))$ and $\Omega_x^-(y) = \text{pr}_-(\Omega_x(y))$.

Lemma 6.12. *Fix $\lambda \in A^{++}$, and $z \in V_{\lambda}(x)$. Then we have $\mu_{x,+}(\Omega_x^+(y)) = \frac{1}{K_1 q^{2\alpha_1(\lambda)}}$ for some constant K_1 depending only on q . Similarly, we have $\mu_{x,-}(\Omega_x^-(y)) = \frac{1}{K_2 q^{2\alpha_2(\lambda)}}$ for some constant K_2 .*

Proof. We treat only the case of $\mu_{x,+}$, the other case being similar. Assume first that $\lambda \neq 0$ satisfies $\alpha_2(\lambda) = 0$ (the result is also true in this case, even so $\lambda \notin A^{++}$). Let $y \in V_{\lambda}(x)$. The point y is on some geodesic ray from x to a vertex of type $+$. Then $\Omega_x^+(y)$ is the set of u such that $y \in [xu]$. If $C \in \text{Ch}(\Delta)$ is a chamber adjacent to u , then this ray belongs to $Q(x, C)$, and in particular $C \in \Omega_x(y)$. Hence $\text{pr}_+^{-1}(\Omega_x^+(y)) = \Omega_x(y)$. So $\mu_{x,+}(\Omega_x^+(y)) = \mu_x(\Omega_x(y)) = \frac{1}{N_{\lambda}}$. In this case, by [50, Theorem 5.15] we have $N_{\lambda} = K_1 q^{2\alpha_1(\lambda)}$ for some constant K_1 , hence the result.

Now let us treat the case of $\lambda \in A^{++}$. Let $z \in V_{\lambda}(x)$ and let $\lambda_1 = \alpha_1(\lambda)t_1$. Let y be the point in $V_{\lambda_1}(x)$ which is in an apartment containing x and z . Note that since $\lambda \in A^{++}$, we have $\lambda_1 \neq 0$, hence $y \neq x$.

Let $u \in \Omega_x^+(y)$. In other words, u is the endpoint of some geodesic starting by the segment $[xy]$. We claim that there exists an apartment containing u, x and z . Indeed, let $y_0 = y, y_1, \dots, y_n, \dots$ be the consecutive vertices of the ray $[yu)$, and let F_n be the combinatorial convex hull of y_n, x and z . We prove by induction that, for every n , F_n is in some sector based at x , and furthermore, seeing α_1 as a linear form on this sector in the obvious way, we have $\alpha_1(y') \leq \alpha_1(y_n)$.

The induction basis being clear, let us show the induction step. Assume that F_n is contained in some apartment A_n . Furthermore, the induction hypothesis implies that F_n is contained in the half-space H delimited by $y_n + \text{Ker}(\alpha_1)$. There exists a simplex adjacent to H which contains y_{n+1} . By Theorem 2.3, the union of H and this chamber is contained in an apartment. It follows that F_{n+1} is also contained in an apartment. Now it is easy to see that it is actually contained in a sector and that $\alpha_1(y') \leq \alpha_1(y_{n+1})$ for every $y' \in F_{n+1}$.

This proves the claim. It follows that there exists some chamber $C \in \Omega_x(z)$ such that $u \in \pi_+(C)$, so that $\Omega_x^+(y) \subset \Omega_x^+(z)$. Since it is clear that $\Omega_x(z) \subset \Omega_x(y)$, we have that $\Omega_x^+(y) = \Omega_x^+(z)$, and the result follows. \square

Lemma 6.13. *Let $x, y \in X^{(0)}$. For all $u \in \Delta_+$ and $v \in \Delta_-$ we have*

$$\frac{d\mu_{x,+}}{d\mu_{y,+}}(u) = q^{2h_u(x,y)} \quad \text{and} \quad \frac{d\mu_{x,-}}{d\mu_{y,-}}(v) = q^{2h_v(x,y)}.$$

Proof. Let $\lambda \in A^+$, and $z \in V_\lambda(x)$. Let $C \in \Omega_x(z)$. By [51, Lemma 3.13] we have, as soon as λ is large enough, $z \in V_{\lambda-h_C(x,y)}(y)$ and $\Omega_x(z) = \Omega_y(z)$. Hence we get $\mu_{y,+}(\Omega_x^+(z)) = \frac{1}{Kq^{2(\alpha_1(\lambda)-h_u(x,y))}}$ while $\mu_{x,+}(\pi_+(\Omega_x(z))) = \frac{1}{Kq^{2\alpha_1(\lambda)}}$. As λ is arbitrary large, this proves the result. \square

Definition 6.14. Let $(u, v) \in \Delta_\pm^{\text{op}}$, and $x \in X^{(0)}$. We define $\beta_x(u, v) = h_u(x, z) + h_v(x, z)$ where z is a point on a geodesic between u and v . This does not depend on the choice of z . We define the measure m_\pm on Δ_\pm^{op} by

$$dm_\pm(u, v) = q^{2\beta_x(u, v)} d\mu_{x,+}(u) d\mu_{x,-}(v)$$

Lemma 6.15. *The measure m_\pm does not depend on the choice of x . It is a Γ -invariant Radon measure on Δ_\pm^{op} .*

Proof. Follows from similar calculations as in §6.2. \square

We now want to investigate more precisely the relationship between m_\pm and m . Our goal is to obtain a preliminary relation between μ_x and m_\pm (Proposition 6.19). We need first to introduce a few more notations.

Let $\Delta_{u,v}$ be the set of chambers C which are adjacent to both a chamber adjacent to u and a chamber adjacent to v . The set of pairs of distinct chambers in $\Delta_{u,v}$ is the preimage of (u, v) by the map $\Delta^{2,\text{op}} \rightarrow \Delta_\pm^{\text{op}}$.

Recall from §2.3 the construction of the panel tree T_u . The topology of the boundary of T_u is generated by the sets $\Omega_x^T(y)$ of endpoints of geodesic rays starting from x passing through y . There is a natural projection $\pi_u : X \rightarrow T_u$, which associates to a point $x \in X$ the class of the geodesic ray $[xu]$. If C is a chamber adjacent to u , then $\pi_u(Q(x, C))$ is a half-line in T_u . The endpoint of this half-line does not depend on x . This defines a map from the set $\text{Res}(u)$ of chambers adjacent to u to the boundary of T_u . This map is an homeomorphism.

Lemma 6.16. *Let $p_{u,v}$ be the restriction of the projection map $\text{proj}_u : \text{Ch}(\Delta) \rightarrow \text{Res}(u)$ to $\Delta_{u,v}$. Then $p_{u,v}$ is a homeomorphism.*

Proof. The inverse image by π_u of a geodesic ray is a half-flat. The boundary of this flat (in X) will consist of three chambers, exactly one of them being in $\Delta_{u,v}$. This proves the bijectivity of $p_{u,v}$.

Let us fix a point $x \in X$, and let $x' = \pi_u(x)$. Fix also $y' \in T_u$ and $C \in p_{u,v}^{-1}(\Omega_x^T(y'))$. Then $\pi_u(Q(x, C))$ is a geodesic ray in T_u passing through y' . So there exists $y \in Q(x, C)$ such that $\pi_u(y) = y'$. Conversely, if $y \in X$ is such that $\pi_u(y) = y'$, let $C \in \Omega_x(y) \cap \Delta_{u,v}$. Then $y \in Q(x, C)$, so that y' is on the geodesic ray from x' to $p_{u,v}(C)$. Hence $p_{u,v}^{-1}(\Omega_{x'}^T(y'))$ is the union of all $\Omega_x(y) \cap \Delta_{u,v}$, where y varies in $\pi_u^{-1}(y')$. So it is open in $\Delta_{u,v}$, which proves that $p_{u,v}$ is continuous. Since $\Delta_{u,v}$ is compact (as an intersection of closed subsets of $\text{Ch}(\Delta)$), $p_{u,v}$ is a homeomorphism. \square

We can define a measure on T_u in a similar way as we defined the measure on $\text{Ch}(\Delta)$. For $x, y \in T_u$, let $\Omega_x^T(y)$ be the set of $\xi \in \partial T_u$ such that y is on the geodesic ray $[x\xi]$. We define the measure μ_x^T by the formula $\mu_x^T(\Omega_x^T(y)) = \frac{1}{K'q^{d(x,y)}}$ valid for every $x \neq y \in T_u$ and for some normalization constant K' (which is actually equal to $\frac{q+1}{q}$).

This allows us to define a measure on $\Delta_{u,v}$, via the map $p_{u,v}$: for $x \in X^{(0)}$, we define the measure $\mu_{u,v;x}$ as $\mu_{u,v;x} = (p_{u,v}^{-1})_* \mu_{\pi_u(x)}^T$.

Lemma 6.17. *Let $y \in V_\lambda(x)$. The measure $\mu_{u,v;x}$ satisfies $\mu_{u,v;x}(\Omega_x(y)) = \frac{1}{K'q^{\ell(\lambda)}}$ whenever $\Omega_x(y) \cap \Delta_{u,v} \neq \emptyset$.*

Proof. Assume that $\Omega_x(y) \cap \Delta_{u,v}$ is not empty. Let $C \in \Omega_x(y) \cap \Delta_{u,v}$, and let $C_1 = \text{proj}_u(C)$. By definition C and C_1 share a vertex of type $-$, say v' . It follows that the geodesic ray $[xv']$ is contained in both $Q(x, C_1)$ and $Q(x, C)$.

Let y' be the point on this geodesic ray such that $\pi_u(y) = \pi_u(y')$. We claim first that

$$p_{u,v}(\Omega_x(y') \cap \Delta_{u,v}) = \Omega_{\pi_u(x)}^T(\pi_u(y)).$$

Indeed, it is clear that $p_{u,v}(\Omega_x(y') \cap \Delta_{u,v}) \subset \Omega_{\pi_u(x)}^T(\pi_u(y))$. Let $C \in p_{u,v}^{-1}(\Omega_{\pi_u(x)}^T(\pi_u(y)))$, and let $C'_1 = \text{proj}_u(C)$. As C'_1 is adjacent to u , the sector $Q(x, C'_1)$ contains the geodesic ray $[xu]$. Furthermore, since $\xi \in \Omega_{\pi_u(x)}^T(\pi_u(y))$, the sector $Q(x, C'_1)$ must contain some point z with $\pi_u(y) = \pi_u(z)$, hence the geodesic ray $[zu]$. But y' is in the convex hull of this geodesic ray and x . This proves that $y' \in Q(x, C'_1)$. Let v'' be the vertex of type $-$ of C'_1 . It follows that $y' \in [xv'']$. Since $[xv''] \subset Q(x, C')$, we also have $y' \in Q(x, C')$, or equivalently, $C' \in \Omega_x(y')$, which proves the claim.

Now, since it is clear that $p_{u,v}(\Omega_x(y) \cap \Delta_{u,v}) \subset \Omega_{\pi_u(x)}^T(\pi_u(y))$, it follows that $\Omega_x(y) \cap \Delta_{u,v} \subset \Omega_x(y') \cap \Delta_{u,v}$. Now let $C_2 = \text{proj}_v(C)$, u' the vertex contained in both chambers C and C_2 , and y'' the point on $[xu']$ such that $\pi_v(y) = \pi_v(y'')$. By a symmetric argument, we have $\Omega_x(y) \cap \Delta_{u,v} \subset \Omega_x(y'') \cap \Delta_{u,v}$. The same argument, replacing y with y' , leaves to the inclusion $\Omega_x(y') \cap \Delta_{u,v} \subset \Omega_x(y'') \cap \Delta_{u,v}$.

So we have $\Omega_x(y') \cap \Delta_{u,v} \subset \Omega_x(y') \cap \Omega_x(y'') \cap \Delta_{u,v}$. Since y is in the convex hull of y' and y'' , we have $\Omega_x(y') \cap \Omega_x(y'') \subset \Omega_x(y)$. Therefore, we have the equality

$$p_{u,v}(\Omega_x(y) \cap \Delta_{u,v}) = \Omega_{\pi_u(x)}^T(\pi_u(y))$$

Hence, we get that $\mu_{u,v;x}(\Omega_x(y)) = \frac{1}{K'q^{d(\pi_u(x), \pi_u(y))}}$. By definition, we have $\pi_u(x) = \pi_u(y')$. Let λ' be such that $y' \in V_{\lambda'}(x)$. In the chamber C it is easy to see that $\ell(\lambda) = \ell(\lambda')$, and looking in the chamber C_1 we see that $d(\pi_u(x), \pi_u(y')) = \ell(\lambda')$. \square

Finally, we need a calculation similar to the one of Lemma 6.2. Let $x \in X^{(0)}$, $\lambda \in A^+$, and fix $y \in V_\lambda(x)$. For $\varepsilon = +$ or $-$, let $Z_\varepsilon(y)$ be the set of $y' \in V_\lambda(x)$ such that there exists some $C \in \Omega_x(y)$ and $C' \in \Omega_x(y')$ which is ε -adjacent to C .

Lemma 6.18. *Assume that $\lambda \in A^{++}$. Then there exists some constants K_+ and K_- such that $|Z_+(y)| = K_+q^{\alpha_1(\lambda)}$ and $|Z_-(y)| = K_-q^{\alpha_2(\lambda)}$.*

Proof. We treat the case of Z_+ . We proceed by induction on $\max(\alpha_1(\lambda), \alpha_2(\lambda))$. Let $\lambda' = \lambda + t_1$ and $\lambda'' = \lambda + t_2$. As in Lemma 6.2, we use the projections $p_1 : V_{\lambda'}(x) \rightarrow V_\lambda(x)$ and $p_2 : V_{\lambda''}(x) \rightarrow V_\lambda(x)$, which associates to a point y the unique point in $V_\lambda(x)$ which is in the convex hull of x and y .

Let $y' \in V_{\lambda'}(x)$ and $y'' \in V_{\lambda''}(x)$ be such that $p_1(y') = p_2(y'')$. Let $y = p_1(y')$. It is sufficient to prove that $Z_+(y')$ has the same cardinality as $Z_+(y)$ and that $|Z_+(y'')| = q|Z_+(y)|$. Note that p_1 (resp. p_2) restricts to a map $p'_1 : Z_+(y') \rightarrow Z_+(y)$ (resp. $p'_2 : Z_+(y'') \rightarrow Z_+(y)$). We prove first that p'_1 is bijective.

Let $y_1 \in Z_+(y)$. By definition there exists $C \in \Omega_x(y)$ and $C_1 \in \Omega_x(y_1)$ which share a vertex of type $+$, say u . Let y'_1 be the first vertex on the geodesic ray $[y_1u]$ after y_1 . Then

$y_1 \in V_\lambda(x)$ satisfies $p_1(y'_1) = y_1$. Furthermore, any $z_1 \in p_1^{-1}(y_1)$ is on this geodesic ray, so that $z_1 = y'_1$. Hence p'_1 is bijective, and we have $|Z_+(y')| = |Z_+(y)| = K_+q^{\alpha_1(\lambda)} = K_+q^{\alpha_1(\lambda')}$ by induction.

Consider now a simplex with $[y_1y'_1]$ as an edge, and let y''_1 be the third vertex of this simplex. Since $\alpha_2(\lambda) > 0$, there is one choice of such a simplex for which $y''_1 \notin V_{\lambda''}(x)$ (the one in the convex hull of y'_1 and x), and q other choices. Each of these other choices is a point in $p_2^{-1}(y_1)$. So $|Z_+(y'')| = q|Z_+(y)| = K_+q^{\alpha_1(\lambda'')}$. \square

Proposition 6.19. *For every $x \in X^{(0)}$ we have*

$$\mu_x = \int \mu_{u,v;x} d\mu_{x,+}(u) d\mu_{x,-}(v).$$

Proof. Let $\tilde{\mu} = \int \mu_{u,v;x} d\mu_{x,+}(u) d\mu_{x,-}(v)$. It suffices to prove that $\tilde{\mu}(\Omega_x(y)) = \frac{1}{Kq^{2\ell(\lambda)}}$ for every $y \in V_\lambda(x)$. In fact, it is sufficient to prove this for every λ large enough. So we may and shall assume that $\lambda \in A^{++}$.

By Lemma 6.17, we have $\mu_{u,v;x}(\Omega_x(y)) = \frac{1}{K'q^{\ell(\lambda)}}$ whenever $\Omega_x(y) \cap \Delta_{u,v} \neq \emptyset$, and 0 if not. So we have to calculate the $\mu_{x,+} \times \mu_{x,-}$ measure of the set D of (u,v) such that $\Omega_x(y) \cap \Delta_{u,v} \neq \emptyset$.

We claim that

$$D = \bigcup_{z_1 \in Z_+(y)} \bigcup_{z_2 \in Z_-(y)} (\Omega_x^+(z_2) \times \Omega_x^-(z_1)) \cap \Delta_{\pm}^{\text{op}}.$$

Let $(u,v) \in D$. Let $C \in \Omega_x(y) \cap \Delta_{u,v}$ and let $C_2 = \text{proj}_u(C)$. By definition C_2 and C have a common vertex of type $-$. Let $z_2 \in Q(x, C_2) \cap V_\lambda(x)$. Then we have $z_2 \in Z_-(y)$, and $u = \pi_+(C_2) \in \Omega_x^+(z_2)$. Similarly we find $z_1 \in Z_+(y)$ such that $v \in \Omega_x^-(z_1)$.

Conversely, let $z_1 \in Z_+(y)$ and $z_2 \in Z_-(y)$. By definition there exists a chamber C_2 in $\Omega_x(z_2)$ which is adjacent to some chamber in $\Omega_x(y)$. Let $u = \pi_+(C_2)$ and $v' = \pi_-(C_2)$. Let $v \in \Omega_x^-(z_1)$ and assume that u and v are opposite. Let $C = \text{proj}_{v'}(v)$ and $u' = \pi_+(C)$. Note that $Q(x, C)$ is the sector whose boundary is the geodesic rays from x to u' and from x to v' . By construction y is in the convex hull of these two geodesic rays. Hence $y \in Q(x, C)$, or in other words, $C \in \Omega_x(y)$, and C is also in $\Delta_{u,v}$, which proves that $(u,v) \in D$.

Thus the claim is proven and we deduce that

$$(\mu_{x,+} \times \mu_{x,-})(D) = \sum_{z_1 \in Z_+(y)} \sum_{z_2 \in Z_-(y)} \mu_{x,+}(\Omega_x^+(z_2)) \mu_{x,-}(\Omega_x^-(z_1)).$$

From Lemma 6.18 we know that $|Z_+(y)| = K_+q^{\alpha_1(\lambda)}$ and $|Z_-(y)| = K_-q^{\alpha_2(\lambda)}$. Furthermore, for every $z \in V_\lambda(x)$, we have $\mu_{x,+}(\Omega_x^+(z)) = \frac{1}{Kq^{2\alpha_1(\lambda)}}$ and $\mu_{x,-}(\Omega_x^-(z)) = \frac{1}{Kq^{2\alpha_2(\lambda)}}$. Hence we find that

$$(\mu_{x,+} \times \mu_{x,-})(D) = K_+q^{\alpha_1(\lambda)} \times K_-q^{\alpha_2(\lambda)} \times \frac{1}{K'_1K'_2q^{2\alpha_2(\lambda)+2\alpha_1(\lambda)}}.$$

It follows that $\tilde{\mu}(\Omega_x(y)) = \frac{1}{K''q^{2\ell(\lambda)}}$ for some constant K'' . In other words $\tilde{\mu}$ is a multiple of μ_x . Since both are probability measures, we have $\tilde{\mu} = \mu_x$. \square

6.4 Measuring Diagram (3)

The following is a general principle which proof is standard and will be omitted.

Lemma 6.20. *Let H, Λ be locally compact groups and $H' < H$ be a subgroup such that there exists an H -invariant measure on H/H' . Let U be a locally compact space on which Λ acts by homeomorphism and ν a Λ -invariant Radon measure on U . Let V be a Λ -invariant H/H' bundle over U , that is V is a locally compact space endowed with $\Lambda \times H$ -action by homeomorphisms and a map $\pi : V \rightarrow U$ which is Λ -equivariant and H -invariant such that for each $u \in U$, $\pi^{-1}(\{u\})$ is H -homeomorphic to H/H' . Then, fixing an H -invariant measure on H/H' and a Borel section $s : U \rightarrow V$ such that for every $u \in U$, $s(u)$ is H' -fixed and defining the orbit map $i_u : H/H' \rightarrow Hs(u)$, the formula*

$$C_c(V) \ni f \mapsto \int_U d\nu \int_{H/H'} f \circ i_u$$

gives a $\Lambda \times H$ -invariant Radon measure on V .

In view of Lemma 5.20 we now use this principle for $U = \Delta_{\pm}^{\text{op}}$, $\nu = m_{\pm}$ and $H = M \times S$ for the various spaces appearing in the chain of spaces and corresponding chain of subgroups H'

$$\mathcal{S} \rightarrow \mathcal{F} \rightarrow \Delta^{2, \text{op}} \rightarrow \Delta_{\pm}^{\text{op}}, \quad \{e\} < \text{Stab}_H(\Sigma) < \text{Stab}_H(\partial\Sigma) < H$$

(see diagram (3) and the discussion after for the definitions of the maps). We obtain Γ -invariant Radon measures on the spaces in the chain, and the maps are clearly measure class preserving. Similarly we define a Γ -invariant Radon measure on \mathcal{S}/S . The following lemma checks compatibility with the measures we have already defined.

Lemma 6.21. *The measure obtained on $\Delta^{2, \text{op}}$ is the class of $\mu \times \mu$.*

Proof. In view of Proposition 6.19, it suffices to prove that for almost every $(u, v) \in \Delta_{\pm}^{\text{op}}$, the measure $\mu_{u,v;x} \times \mu_{u,v;x}$ is in the same measure class as the Haar measure on $H/\text{Stab}_H(\partial\Sigma)$. Fix $(u, v) \in \Delta_{\pm}^{\text{op}}$. Then H acts on the $(q+1)$ -regular tree $T_{u,v}$. The action on the boundary $\partial T_{u,v}$ is 2-transitive. The stabilizer of a pair of distinct points $(\xi, \xi') \in \partial T_{u,v} \times \partial T_{u,v}$ is exactly $H' = \text{Stab}_H(\partial\Sigma)$. Hence the Haar measure on H/H' is the invariant measure $m_{u,v}$ on the set of distinct pairs of points in the boundary of $T_{u,v}$. This measure, as can be checked with similar computations as in §6.2, is given by the formula $dm_{u,v}(\xi, \xi') = q^{-\beta_x^T(\xi, \xi')} d\mu_x^T(\xi) d\mu_x^T(\xi')$ (where $\beta_x^T(\xi, \xi') = (\xi, \xi')_x$ is the usual Gromov product on T , and μ_x^T is the measure defined before Lemma 6.17). Hence it is in the same class as $\mu_x^T \times \mu_x^T$, which is identified (when viewed via $p_{u,v}$ as a measure on $\text{Ch}(\Delta)$) to $\mu_{u,v;x} \times \mu_{u,v;x}$. \square

Remark 6.22. It is possible, by a more precise calculation, in the spirit of Lemma 6.17, to prove that η is actually the same measure as m . This also follows from Theorem 7.1 below, as the ergodicity of the Γ -action on $\Delta^{2, \text{op}}$ implies that there exists at most one Γ -invariant Radon measure in the class $\mu \times \mu$, thus that η coincides with m (up to a factor which is easily checked to be 1).

Note that $\text{Stab}_H(\Sigma)$ is normal in $\text{Stab}_H(\partial\Sigma)$ and the quotient group is naturally identified with A . It is easy to deduce that the Radon measure on \mathcal{F} is A -invariant. In view of Lemma 5.3 and Theorem 5.21 we obtain the following.

Corollary 6.23. *There exists a $M \times S$ -invariant probability measure on \mathcal{S}/Γ and A -invariant probability measure on \mathcal{F}/Γ such that $\mathcal{S}/\Gamma \rightarrow \mathcal{F}/\Gamma$ is measure preserving.*

In the above we explained how to construct invariant Radon measures on some spaces. Focusing now on measure classes merely, we observe that \mathcal{S} is a source in diagram (3) (for every other space in the diagram there is a map from \mathcal{S} to the space), thus pushing the measure class from \mathcal{S} to each other space and using Lemma 6.21 and the commutativity of the diagram we obtain the following corollary.

Corollary 6.24. *There is a unique way to obtain measure classes on each space in diagram (3) such that all maps are measure class preserving and it is compatible with all the measure classes we have considered on various spaces so far.*

7 Ergodicity

In this section we establish the ergodicity properties required by the proof of the main result. For the sake of readability, we often omit the measures from our notations. All measures are understood to be those defined in §6, see Corollary 6.24. Similarly, unspecified maps are those discussed in §5.4, see Diagram (3).

7.1 Ergodicity the Cartan flow

This subsection is devoted to the proof of the following theorem.

Theorem 7.1. *The following two actions are ergodic:*

- The Γ -action on $\Delta^{2,op}$.
- The A -action on \mathcal{F}/Γ .

It is possible to use [25] to prove that $(\text{Ch}(\Delta), \mu)$ is the Poisson boundary of Γ (endowed with a suitable probability measure), at least in the case when Γ acts transitively on vertices in X . Using the fact that the action of a group is always ergodic on the square of its Poisson boundary, we get directly that the ergodicity of the Γ action on $\text{Ch}(\Delta) \times \text{Ch}(\Delta)$. We choose to present another proof, which is based on the classical ‘‘Hopf argument’’, see Appendix B. In order to facilitate the readability of the proof, we summarize some of the constructions and discussions given in the previous section in the following diagram, which should be compared with the diagram (7) appearing in Theorem B.1.

$$\begin{array}{ccc}
 & \mathcal{F} & \\
 \phi \swarrow & & \searrow \psi \\
 \mathcal{F}/\Gamma & & \mathcal{F}/A \simeq \Delta^{2,op} \\
 & \swarrow p_1 & \searrow p_2 \\
 & \text{Ch}(\Delta) & \text{Ch}(\Delta)
 \end{array} \tag{4}$$

Proof of Theorem 7.1. Set $T = A$. Note that this is an abelian, hence amenable group. Set also $\Lambda = \Gamma$ and $Z = \mathcal{F}$. Note that Z is a locally compact topological space endowed with commuting actions of T and Λ and both actions (separately) $\Lambda \curvearrowright Z$ and $T \curvearrowright Z$ are proper. Set $V = Z/\Lambda$ and $Y = Z/T$ and let $\phi: Z \rightarrow X$ and $\psi: Z \rightarrow Y$ be the quotient maps. By Lemma 5.3, the space V is compact. By Lemma 5.4, we have $Y \simeq \Delta^{2,\text{op}}$. We let $W_1 = W_2 = \Delta$ and let π_1 and π_2 be the first and second coordinate projections $p_1, p_2: \Delta^{2,\text{op}} \rightarrow \Delta$ correspondingly. By the fact that the Radon measure m is equivalent to the product measure $\mu \times \mu$ we infer that $\pi_1^*(L^\infty(\Delta)) \cap \pi_2^*(L^\infty(\Delta))$ consists only of the essentially constant functions.

Recall that $Z = \mathcal{F}$ is the set of simplicial embeddings from Σ to X where a basis for the topology on \mathcal{F} is given by the sets of all maps which are the same in restriction to a ball around 0. Let t_1 be a vector of $T = A$ contained in the interior of the positive Weyl chamber. It is of course central, by the commutativity of A . Fix $z, z' \in \mathcal{F}$ such that $\pi_2\psi(z) = \pi_2\psi(z')$. The latter equality means that the maps z and z' coincide at infinity of the negative Weyl chamber. Thus $z(\Sigma)$ and $z'(\Sigma)$ have a common negative subsector, and there exist $t, t' \in T$ such that $z|_{Q^- - t} = z'|_{Q^- - t'}$. Equivalently we write $tz|_{Q^-} = t'z'|_{Q^-}$. Since t_1 is in the interior of the positive Weyl chamber, we see that for every $\sigma \in \Sigma$ we have for every n large enough, $\sigma - nt_1 \in Q^-$. It follows that for every compact neighborhood B of $0 \in \Sigma$, for every n large enough, $B - nt_1 \in Q^-$ and therefore, $tz|_{B - nt_1} = t'z'|_{B - nt_1}$. Equivalently we write $t_1^n tz|_B = t_1^n t'z'|_B$. It clearly follows that the two sequences $(t_1^n t\phi(z))_{n \in \mathbb{N}}$ and $(t_1^n t'\phi(z'))_{n \in \mathbb{N}}$ are proximal in X . We note also that the measure \bar{m} on Z satisfies the assumptions of Theorem B.1. It follows by Theorem B.1 that the A -action on \mathcal{F}/Γ and the Γ -action on $\Delta^{2,\text{op}}$ are ergodic. \square

The following corollary is straightforward, as Δ_\pm^{op} is a Γ -equivariant factor of $\Delta^{2,\text{op}}$.

Corollary 7.2. Γ acts ergodically on Δ_\pm^{op} .

7.2 Ergodicity of the singular Cartan flow

Our goal in this subsection is to prove the following.

Theorem 7.3. The Γ -action on \mathcal{S}/S is ergodic.

In the sequel we denote by Y the space \mathcal{S}/S . It is equipped with the measure class pushed from \mathcal{S} . Moreover $M \times \Gamma$ acts on Y and preserves this measure class. Note that the continuous surjective $\mathcal{S} \rightarrow \Delta_\pm^{\text{op}}$ factors through Y by Lemma 5.20. For $(u, v) \in \Delta_\pm^{\text{op}}$ we let $Y(u, v)$ be the associated fiber under the map $Y \rightarrow \Delta_\pm^{\text{op}}$.

Recall from Definition 5.8 the definitions of the maps $\pi_+ : \widetilde{\mathcal{F}} \rightarrow \overline{\Delta}_+$ and $\pi_- : \widetilde{\mathcal{F}} \rightarrow \overline{\Delta}_-$. Restricting these maps to \mathcal{S} and noting that they factor through Y , we abuse the notation and consider these maps as $\pi_+ : Y \rightarrow \overline{\Delta}_+$ and $\pi_- : Y \rightarrow \overline{\Delta}_-$. In order to meet the notation introduced in Appendix B we also denote $W_+ = \overline{\Delta}_+$ and $W_- = \overline{\Delta}_-$. We endow these spaces with the measure classes pushed under π_+, π_- .

Let p_+ (resp. p_-) be the second coordinate of π_+ (resp. π_-). In other words, $p_+(\phi)$ is the map $f : T \rightarrow T_{\phi_+}$ which is defined as $f(x) = \phi(x, t)$ for t large enough. The space Y_u (resp. Y_v) is defined as $p_-(Y(u, v))$ (resp. $p_+(Y(u, v))$). Since Y is closed under positive and negative equivalences, we have bijections $Y(u, v) \rightarrow Y_u$ and $Y(u, v) \rightarrow Y_v$. By composition we have a bijection $Y_u \rightarrow Y_v$ for every $(u, v) \in \Delta_\pm^{\text{op}}$.

Note that the space $Y(u, v)$ is in bijection with M . More precisely, it is the M -orbit of some (or any) $\phi \in Y(u, v)$, by Lemma 5.20. Fix $\phi \in Y(u, v)$, and let $O_\phi : M \rightarrow Y(u, v)$ be the orbit map $a \mapsto a\phi$. We define $\lambda(u, v) = (O_\phi)_*(\text{Haar}(M))$.

Similarly, each space Y_u is the M -orbit of some element (with the natural action by precomposition) and we equip the space Y_u with the Haar measure coming from this identification, which we denote by λ_u .

Lemma 7.4. *The measures $\lambda(u, v)$ and λ_u are independent on the choice of ϕ . Furthermore, we have $\psi_*\lambda(u, v) = \lambda_u$, where $\psi : Y(u, v) \rightarrow Y_u$ is as above.*

Proof. Let us argue first that $\lambda(u, v)$ is independent of the choice of $\phi \in Y(u, v)$. Let ϕ, ϕ' be elements of $Y(u, v)$. We know that there exists an $a \in M$ such that $\phi' = a\phi$. So, for any $m \in M$, we have $m\phi' = ma\phi$. Let $R_a : M \rightarrow M$ be the right translation by a . This means that $O_{\phi'} = O_\phi \circ R_a$. By Theorem 5.21 M is unimodular thus we conclude that $(O_\phi)_*(\text{Haar}(M)) = (O_{\phi'})_*(\text{Haar}(M))$. The argument is similar for λ_u .

Now let us prove that $\psi_*\lambda(u, v) = \lambda_u$. We identify $Y_{u,v}$ to M by the choice of some $\phi \in Y_{u,v}$. Using the previous argument, we can choose to identify M to the orbit of $\psi(\phi) \in Y_u$. With these identifications, ψ is the identity map, which of course preserves the Haar measure. \square

In the sequel we will have to consider some full measure subsets of various spaces. Let us introduce some useful vocabulary first. We fix a subset $D \subset Y$.

Let (u_1, u_2, \dots, u_n) be a sequence of consecutively opposite vertices, with $u_1 \in \Delta_-$. Consider a sequence $\phi_1, \dots, \phi_{n-1}$ of elements of D such that $\phi_i^- = u_i$, $\phi_i^+ = u_{i+1}$ and ϕ_1 is positively equivalent to ϕ_2 , ϕ_2 is negatively equivalent to ϕ_3 , and so on. To such a sequence we associate the map $p_+(\phi_{n-1}) \circ p_-(\phi_1)^{-1}$ if n is even, $p_-(\phi_{n-1}) \circ p_-(\phi_1)^{-1}$ if n is odd. This is a map from T_{u_1} to T_{u_n} . We denote by $D(u_1, u_2, \dots, u_n)$ the set of maps from T_{u_1} to T_{u_n} that can be obtained this way, when the sequence (ϕ_1, \dots, ϕ_n) varies.

We note that the set $Y(u, v)$ is the same as the one defined above. Furthermore, the set $Y(u_1, \dots, u_n)$ is again an M -orbit of any element. By a similar argument as in Lemma 7.4, we can define a measure $\lambda(u_1, \dots, u_n)$ as the image of the Haar measure on M by this identification.

Definition 7.5. A pair (u, v) of opposite vertices is **D -full** if the set $D \cap Y(u, v)$ is of full $\lambda_{u,v}$ -measure in $Y(u, v)$.

A sequence (u_1, u_2, \dots, u_n) of consecutively opposite vertices is **D -full** if the set $D(u_1, u_2, \dots, u_n)$ is of full $\lambda(u_1, \dots, u_n)$ -measure in the set $Y(u_1, u_2, \dots, u_n)$.

Lemma 7.6. *If (u_1, \dots, u_n, v) is D -full, and (v, v_1, \dots, v_m) is also A -full, then the sequence $(u_1, \dots, u_n, v, v_1, \dots, v_m)$ is D -full.*

Proof. An element of $D(u_1, \dots, u_n, v)$ can be described by a sequence $\phi_1, \dots, \phi_{n-1}, \phi_n$ as above, where in particular ϕ_n goes from u_n to v . Since we assume (u_1, \dots, u_n, v) to be D -full, the set of maps ϕ_n that can be the last element of such a sequence is of full measure in $Y(u_n, v)$. So (by Lemma 7.4) its projection to Y_v is also of full measure. Similarly, the set $D(v, v_1, \dots, v_m)$ is described by sequences $\phi'_0, \phi'_1, \dots, \phi'_{m-1}$, and the set of possible ϕ_0 is of full measure in $Y(v, v_1)$. So its projection is again of full measure in Y_v .

So we have two full measure sets of Y_v . It follows that their intersection is again of full measure in Y_v . If an element is in this intersection, then there are ϕ_n and ϕ'_0 as above which are positively equivalent. Hence the sequence $(\phi_1, \dots, \phi_n, \phi'_0, \dots, \phi'_m)$ will define an element of $D(u_1, \dots, v, v_1, \dots, v_m)$. It follows that the sequence (u_1, \dots, v_m) is D -full. \square

Lemma 7.7. *Assume that D has full measure in Y , and let $n \geq 2$. Then $\nu^{\otimes n}$ -almost every sequence (u_1, \dots, u_n) is D -full.*

Proof. Recall that the set Y is identified to $\Delta_- \times \Delta_+ \times M$, and the measure is in the same class as the product measure. Using Fubini, the fact that D is of full measure means that for almost every u, v , the set of elements in $D(u, v)$ is of full measure in M . This proves the Lemma for $n = 2$.

Combining Lemma 7.6 with the fact that almost every sequence of length 2 is D -full, we conclude that almost every sequence is D -full. \square

Definition 7.8. Let $u \in \Delta_-$. We define the subgroup $M_D(u)$ of M as the group generated by all perspectivities of the form $[u; u_1; \dots; u_n; u]$ where the sequence (u, u_1, \dots, u_n, u) is D -full, and maps of the form $[\gamma : u; v_1; v_2; \dots; v_n, u]$, where the sequence $(\gamma u, v_1, \dots, v_n, u)$ is D -full.

By Lemma 5.20 the group M is the closure of the group M' which is described in Lemma 5.16. We observe that $M' = M_Y(u)$ for all u . Thus for every D and u , $M_D(u)$ is a subgroup of M .

Lemma 7.9. *Assume that D has full measure in Y . Then, for almost every $u \in \Delta_-$, the group $M_D(u)$ is dense in M .*

Proof. For any $n \geq 0$, for almost all u , the set of v such that (u, v) is A -full is of full measure, and the set of v such that (v, v_1, \dots, v_n, u) such is D -full is also of full measure.

It follows that for a fixed (odd) n , for almost every u , the set of (v_1, \dots, v_n) such that (u, v_1, \dots, v_n) is D -full is of full measure. A countable intersection of full measure sets is again of full measure, so that for almost all u , for all n , the set of (v_1, \dots, v_n) such that (u, v_1, \dots, v_n, u) is D -full is of full measure.

Let us fix some u in this full measure set, and prove that $M_D(u)$ is dense. For any fixed n , the set of (v_1, \dots, v_n) such that (u, v_1, \dots, v_n, u) is D -full is of full measure. The measure ν has full support, so this set is dense in $\Delta_+ \times \Delta_- \times \dots \times \Delta_+$. By Proposition 5.12, the set of perspectivities of length $n + 2$ which are in $M_D(u)$ is dense in all perspectivities of length $n + 2$ starting and ending at u .

A similar argument proves that for every $\gamma \in \Gamma$, the set of (v_1, \dots, v_n) such that the sequence $(\gamma u, v_1, \dots, v_n, u)$ is D -full is dense (for almost every u). Taking intersection over (the countable set of) all γ and n , we see that for almost every u , the set of generators of $M_D(u)$ is dense in the set of generators of M' (provided by Lemma 5.16). By Lemma 5.20 it follows that $M_D(u)$ is dense in M , for almost every u . \square

Finally we are ready for the proof of the main theorem of this section. In order to facilitate the readability of the proof, we summarize some of the constructions and discussions given in the previous section in the following diagram, which should be compared with the diagram (7) appearing in Theorem B.1.

$$\begin{array}{ccc}
& \mathcal{S} & \\
\phi \swarrow & & \searrow \psi \\
\mathcal{S}/\Gamma & & \mathcal{S}/S \simeq \\
& \pi_+ \swarrow & \searrow \pi_- \\
& \overline{\Delta}_+ & \overline{\Delta}_-
\end{array} \tag{5}$$

Proof of Theorem 7.3. We want to apply Theorem B.1, putting $Z = \mathcal{S}$, $Y = \mathcal{S}/S$ and $W_+ = \overline{\Delta}_+$, $W_- = \overline{\Delta}_-$ (with the maps π_- and π_+ as described above).

Let us start by checking the proximality condition. Let $z, z' \in \mathcal{S}$ which have the same image in W_+ (the argument is similar for W_-). It follows from the definition of the map to W_+ that, up to replacing z by some translate sz with $s \in S$, we may (and shall) assume that z and z' are positively equivalent (see Definition 5.13). In other words, there exists t_0 such that $z(t, x) = z'(t, x)$ for every $t > t_0$ and every x . We use multiplicative notation for elements of S and denote by s the positive generator of S . It follows that $s^\alpha z(t, x) = s^\beta z'(t, x)$ for some α, β and every $t > 0$. Let B_k be the ball centered at the origin of radius k in $T \times \mathbf{R}$. Then we see that for every m large enough (depending on k), we have $s^m s^\alpha z|_{B_k} = s^m s^\beta z'|_{B_k}$. This proves that the sequences $(s^m s^\alpha z)_{m \in \mathbf{N}}$ and $(s^m s^\beta z')_{m \in \mathbf{N}}$ are proximal in \mathcal{S} .

The only condition left to check in order to apply Theorem B.1 and conclude the proof is that every function class in $L^\infty(Y)$ which is Γ -invariant and is a pullback of functions on W_+ and W_- is essentially constant. We will do it by showing that every such function class is in a pull back under the factor map $Y \rightarrow \Delta_\pm^{\text{op}}$ of a Γ -invariant function class on Δ_\pm^{op} . Indeed, by Corollary 7.2, Γ acts ergodically on the latter space, hence every such function class must be constant.

We now fix a function class f which is Γ -invariant and is a pullback of functions on W_+ and W_- and argue to show that f is constant on a.e fiber of $Y \rightarrow \Delta_\pm^{\text{op}}$, that is for almost every $(u, v) \in \Delta_\pm^{\text{op}}$, f is essentially constant on $Y(u, v)$. Note that by Fubini Theorem f is well defined for a.e such (u, v) and denote its restriction to $Y(u, v)$ by $f_{u,v}$. As usual, we identify $Y(u, v)$ with M and view $f_{u,v}$ as a function on M , a.e defined with respect to the Haar measure.

We let D be the subset of Y on which the two pullbacks of f_+ and f_- agree. Then D is a Γ -invariant full measure subset of Y . We view f as an actual function defined on D . Since f is the pullback of f_+ and f_- , we see that if $\phi \sim_+ \phi'$ and $\phi, \phi' \in D$, then $f(\phi) = f(\phi')$. Let (u, v_1, \dots, v_n, u) be a D -full sequence, and $m = [u, v_1, \dots, v_n, u]$ the associated perspetivity. Then the set of all $\phi \in D(u, v)$ such that $\phi \circ m \in D(u, v)$ is of full measure. If ϕ is an element of this set, then $f_{u,v}(m\phi) = f_{u,v}(\phi)$. So $f_{u,v}$ is invariant by m . Similarly, using also the Γ -invariance of D , if $\gamma \in \Gamma$ and $(\gamma u, v_1, \dots, v_n, u)$ is D -full, then $f_{u,v}$ is essentially $[\gamma : u; v_1; \dots; v_n; u]$ -invariant. It follows that $f_{u,v}$ is $M_D(u)$ invariant. For almost every (u, v) , we know by Lemma 7.7 and Lemma 7.9 that (u, v) is D -full and that $M_D(u)$ is dense in M . Since $f_{u,v}$ is a function in $L^\infty(Y(u, v)) \simeq L^\infty(M)$ which is invariant by $M_D(u)$, $M_D(u)$ is dense in M and the action of M on $L^\infty(M)$ is weak*-continuous, it follows that $f_{u,v}$ is invariant by M , hence constant. \square

8 Non-linearity of Γ

In this section we finally prove Theorem 1.5. We will do so in §8.2. Our proof relies on Bader–Furman’s theory of algebraic representations that will be recalled in §8.1.

8.1 Algebraic Representations

In this section we review Bader–Furman’s theory of gates and algebraic representations, about which the main reference is [5].

Let Γ be a countable group and k be a local field. We fix a representation $\rho: \Gamma \rightarrow \mathbf{G}(k)$, where \mathbf{G} is a k -simple k -algebraic group, and assume that its image is Zariski-dense.

Let Y be a standard Borel space, equipped with a measure μ . We assume that Γ has a measure-class preserving action on Y . Furthermore, a standing assumption is that the Γ -action is moreover ergodic.

Definition 8.1. An **algebraic representation** of Y is a pair (\mathbf{V}, ϕ) , where

- \mathbf{V} is a k -algebraic variety endowed with a k -algebraic action of \mathbf{G} ,
- $\phi: Y \rightarrow \mathbf{V}(k)$ is a measurable map such that $\phi(\gamma y) = \rho(\gamma)y$ for every $\gamma \in \Gamma$ and a.e $y \in Y$.

A **morphism** from a representation (\mathbf{U}, ϕ_U) to (\mathbf{V}, ϕ_V) is a k -algebraic \mathbf{G} -equivariant map $\psi: \mathbf{U} \rightarrow \mathbf{V}$ such that $\phi_U \circ \psi = \phi_V$.

We sometimes abuse notation and speak of the algebraic representation \mathbf{V} instead of (\mathbf{V}, ϕ) .

Theorem 8.2. *Let \mathbf{V} be a k -algebraic variety endowed with a k -algebraic action of \mathbf{G} . Then the orbits of the $\mathbf{G}(k)$ -action on $\mathbf{V}(k)$ are locally closed for the k -topology on $\mathbf{V}(k)$.*

Proof. This is proved in [75, Proposition 3.3.1] under the assumption that the characteristic of k is 0. The general case is proved [9]. For a complete discussion see [4, Proposition 2.2]. \square

Let (\mathbf{V}, ϕ) be an algebraic representation of Y . It follows from Theorem 8.2 and the ergodicity assumption that the pushforward of μ is supported on a unique \mathbf{G} -orbit in \mathbf{V} , hence on a variety which is a quotient of \mathbf{G} .

Now consider the set of all representations of Y into quotients of \mathbf{G} . More precisely, consider the set of all algebraic subgroups $\mathbf{H} < \mathbf{G}$ such that there exists an algebraic representation $(\mathbf{G}/\mathbf{H}, \phi)$. By noetherianity, there exists a minimal element in this set of subgroups.

The above argument is the first step in the proof of the following theorem, which can be found in [5, Theorem 5.3].

Theorem 8.3. *There exists an algebraic subgroup $\mathbf{H} < \mathbf{G}$, and an algebraic representation $\phi: Y \rightarrow \mathbf{G}/\mathbf{H}$, such that for every algebraic representation \mathbf{V} of Y , there exists a morphism of algebraic representations $\mathbf{G}/\mathbf{H} \rightarrow \mathbf{V}$.*

Definition 8.4. The algebraic representation $(\mathbf{G}/\mathbf{H}, \phi)$ is called the **gate** of Y .

The morphism in Theorem 8.3 is in fact uniquely defined. From this, we deduce the following.

Proposition 8.5. *Let $(\mathbf{G}/\mathbf{H}, \phi)$ be the gate of Y . Let M be a Polish group with a Borel action on Y which quasi-preserved μ and commutes with the action of Γ . There exists a continuous homomorphism $M \rightarrow N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}(k)$ which turns the gate map $Y \rightarrow \mathbf{G}/\mathbf{H}$ into a M -equivariant map.*

Proof. The existence of this homomorphism $\psi: M \rightarrow N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}(k)$ is proved in [5, Theorem 6.1]. The only thing left to check is its continuity.

To simplify the notations, let $V = \mathbf{G}/\mathbf{H}(k)$ and $L = N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}(k)$. Let $U = L^0(Y, V)$ be the set of measurable functions from Y to V , with the topology of convergence in measure. Then the space U is endowed with an action of L by post-composition, and an action of M by precomposition. By Theorem 8.2, the L -orbits on U are locally closed. The stabilizer in L of any $f \in U$ is trivial. Hence, for every f , the map $L \rightarrow U$ mapping l to $l.f$ is a homeomorphism.

Let $\phi: Y \rightarrow V$ be the gate map, and $g \in M$. Since the natural map $M \times Y \rightarrow V$ is Borel, by [48, Lemma 7.1.3], we see that the map $M \rightarrow L^0(Y, V)$ which associates to g the map $g.\phi$ is Borel. By equivariance, $\psi(g)$ is precisely the element $l \in L$ such that $g.\phi = l.\phi$. Identifying the L -orbit of ϕ with L proves that ψ is a Borel map from $\text{Aut}(Y)$ to L . Hence we obtain that $\psi: M \rightarrow L$ is Borel. Since it is a homomorphism, it is also continuous [59, Lemma 2.1]. \square

Proposition 8.6. *Assume $Y \rightarrow Z$ is a factor map. Let $\phi_Y: Y \rightarrow \mathbf{G}/\mathbf{H}_Y(k)$ and $\phi_Z: Z \rightarrow \mathbf{G}/\mathbf{H}_Z(k)$ be the corresponding gate maps. Then there exists a canonical k -defined algebraic \mathbf{G} -map $\psi: \mathbf{G}/\mathbf{H}_Y \rightarrow \mathbf{G}/\mathbf{H}_Z$ such that $\phi_Z = \psi \circ \phi_Y$.*

Proof. The existence of ψ follows by considering the composition $Y \rightarrow Z \rightarrow \mathbf{G}/\mathbf{H}_Z(k)$ as a representation of Y and the fact that ϕ_Y is an initial object. \square

8.2 Proof of Theorem 1.5

In this section we fix a local field k , a connected adjoint k -simple k -algebraic group \mathbf{G} and a group homomorphism $\Gamma \rightarrow \mathbf{G}(k)$ with an unbounded and Zariski dense image. We will argue to show that X is Bruhat–Tits.

We will consider the Γ -spaces $\text{Ch}(\Delta)$, $\Delta^{2, \text{op}}$ and \mathcal{S}/S . These spaces are endowed with natural measure classes as discussed in §6 (see in particular §6.1 and Corollary 6.24). We will fix these measure classes, but make them implicit in our notation. By Theorem 7.1 and Theorem 7.3 all these spaces are Γ -ergodic. Thus we may consider their algebraic gates as discussed in §8.1. We consider the map $\mathcal{S}/S \rightarrow \Delta^{2, \text{op}}$ discussed in §5.4 and its $N \rightarrow \{1, \tau\}$ -equivariancy discussed in Proposition 5.23. We also consider the projections $p_1, p_2: \Delta^{2, \text{op}} \rightarrow \text{Ch}(\Delta)$. We apply to the above the discussion of §8.1 and summarize our situation in the following diagram:

$$\begin{array}{ccccc}
N < M^+ & \xrightarrow{\quad} & \mathcal{S}/S & \longrightarrow & \mathbf{G}/\mathbf{H}_{\mathcal{S}/S}(k) & \xrightarrow{\quad} & N_{\mathbf{G}}(\mathbf{H}_{\mathcal{S}/S})/\mathbf{H}_{\mathcal{S}/S}(k) \\
\downarrow & \searrow & \downarrow & \downarrow & \downarrow & \searrow & \downarrow \\
\{1, \tau\} & \xrightarrow{\quad} & \Delta^{2,op} & \longrightarrow & \mathbf{G}/\mathbf{H}_{\Delta^{2,op}}(k) & \xrightarrow{\quad} & N_{\mathbf{G}}(\mathbf{H}_{\Delta^{2,op}})/\mathbf{H}_{\Delta^{2,op}}(k) \\
& & \downarrow p_1 \quad \downarrow p_2 & & \downarrow & & \\
& & \text{Ch}(\Delta) & \longrightarrow & \mathbf{G}/\mathbf{H}_{\text{Ch}(\Delta)}(k) & &
\end{array} \tag{6}$$

Here the spaces on the right-hand side are the gates of the corresponding spaces on the left-hand side and the maps between them are the ones guaranteed by Proposition 8.6. The dotted arrows are group homomorphisms as guaranteed by Proposition 8.5.

We break the proof of Theorem 1.5 into the following four steps:

1. $\mathbf{H}_{\text{Ch}(\Delta)}$, $\mathbf{H}_{\Delta^{2,op}}$ and $\mathbf{H}_{\mathcal{S}/S}$ are proper subgroups of \mathbf{G} .
2. The homomorphism $\{1, \tau\} \rightarrow N_{\mathbf{G}}(\mathbf{H}_{\Delta^{2,op}})/\mathbf{H}_{\Delta^{2,op}}(k)$ is non-trivial.
3. The homomorphism $M^+ \rightarrow N_{\mathbf{G}}(\mathbf{H}_{\mathcal{S}/S})/\mathbf{H}_{\mathcal{S}/S}(k)$ is non-trivial.
4. X is Bruhat–Tits.

Proof of Step 1: Clearly it is enough to show that $\mathbf{H}_{\text{Ch}(\Delta)} \neq \mathbf{G}$, and by the universal property of the gate, it suffices to prove that there exists some algebraic representation $\text{Ch}(\Delta) \rightarrow \mathbf{G}/\mathbf{H}(k)$ with $\mathbf{H} \neq \mathbf{G}$. By the main result of [47] (see also the earlier reference [55, §4.2] for the special case of a lattice Γ acting sharply transitively on the vertices of X), the action of Γ on $\text{Ch}(\Delta)$ is topologically amenable. It follows that the Γ -action on $(\text{Ch}(\Delta), \mu)$ is also amenable in the sense of Zimmer [3, Proposition 3.3.5]. By [4, Theorem 6.1], either there exists an algebraic representation of $\text{Ch}(\Delta) \rightarrow \mathbf{G}/\mathbf{H}(k)$, with \mathbf{H} a proper subgroup of \mathbf{G} , or there exists a Γ -equivariant map to a metric space Z with a proper isometric action of $\mathbf{G}(k)$. The second option is excluded by [36, Theorem 1.1], as $(\text{Ch}(\Delta) \times \text{Ch}(\Delta), \mu \times \mu)$ is Γ -ergodic. \square

Proof of Step 2: If the image of τ in $N_{\mathbf{G}}(\mathbf{H}_{\Delta^{2,op}})/\mathbf{H}_{\Delta^{2,op}}(k)$ were trivial then, by the commutativity of the diagram (6), the two maps $\text{Ch}(\Delta) \times \text{Ch}(\Delta) \xrightarrow{p_1} \text{Ch}(\Delta) \rightarrow \mathbf{G}/\mathbf{H}_{\text{Ch}(\Delta)}(k)$ and $\text{Ch}(\Delta) \times \text{Ch}(\Delta) \xrightarrow{p_2} \text{Ch}(\Delta) \rightarrow \mathbf{G}/\mathbf{H}_{\text{Ch}(\Delta)}(k)$ would coincide. This can happen if and only if these maps are essentially constant, as the two factors $\text{Ch}(\Delta) \times \text{Ch}(\Delta) \xrightarrow{p_1} \text{Ch}(\Delta)$ and $\text{Ch}(\Delta) \times \text{Ch}(\Delta) \xrightarrow{p_2} \text{Ch}(\Delta)$ are obviously independent. But the Zariski closure of the support of the image of $\text{Ch}(\Delta)$ in $\mathbf{G}/\mathbf{H}_{\text{Ch}(\Delta)}(k)$ is Γ -invariant, hence \mathbf{G} -invariant, as $\rho(\Gamma)$ is Zariski-dense, hence it is all of the homogeneous space $\mathbf{G}/\mathbf{H}_{\text{Ch}(\Delta)}$. This contradicts the fact that $\mathbf{H}_{\text{Ch}(\Delta)}$ is a proper subgroup of \mathbf{G} established in the previous step. \square

Proof of Step 3: By Theorem 5.21 the group M^+ is topologically simple. By Proposition 8.5 the homomorphism $M^+ \rightarrow N_{\mathbf{G}}(\mathbf{H}_{\mathcal{S}/S})/\mathbf{H}_{\mathcal{S}/S}(k)$ is continuous. Thus, in order to prove its non-triviality it is enough to exhibit a single element $n \in M^+$ having a non-trivial image. By Proposition 5.23 the homomorphism $N \rightarrow \{1, \tau\}$ is surjective. We let $n \in N < M^+$

be a preimage of τ and observe that if its image in $N_{\mathbf{G}}(\mathbf{H}_{\mathcal{S}/S})/\mathbf{H}_{\mathcal{S}/S}(k)$ were trivial then the image of τ in $N_{\mathbf{G}}(\mathbf{H}_{\Delta^{2,op}})/\mathbf{H}_{\Delta^{2,op}}(k)$ would also be trivial by the commutativity of the diagram (6). \square

Proof of Step 4: The representation $M^+ \rightarrow N_{\mathbf{G}}(\mathbf{H}_{\mathcal{S}/S})/\mathbf{H}_{\mathcal{S}/S}(k)$ is a non-trivial continuous linear representation of M^+ over a local field. We conclude by Theorem 5.21 that X is Bruhat–Tits. \square

8.3 Proof of Theorem 1.1

Let X be a locally finite \tilde{A}_2 -building and Γ be a lattice in $\text{Aut}(X)$. Assume that X is not isomorphic to the building associated to $\text{PGL}_3(D)$, with D a finite dimensional division algebra over a local field. By [49], the group Γ satisfies Kazhdan’s property (T). Observe that every finite index subgroup $\Gamma_0 < \Gamma$ is also a lattice in $\text{Aut}(X)$. By Theorem 1.5 every finite index subgroup of $\Gamma_0 < \Gamma$ has no unbounded Zariski dense representations into $\mathbf{G}(k)$, where k is a local field and \mathbf{G} is a connected adjoint k -simple k -algebraic group. Therefore, the required conclusion follows from Corollary A.3. \square

A On the linearity of finitely generated groups

In this appendix we present various characterizations of the existence of a finite-dimensional linear representation with non-amenable image for general finitely generated groups. The proof is based on a trick invented by Tits in his proof of the so called Tits alternative, and on its strengthening due to Breuillard and Gelander.

Theorem A.1. *For a finitely generated group Γ the following conditions are equivalent.*

- (1) *There exists a commutative ring R with unity, a finitely generated projective R -module M and a group homomorphism $\Gamma \rightarrow \text{Aut}_R(M)$ whose image is not virtually solvable.*
- (2) *There exists a commutative ring R with unity, an integer $n > 0$ and group homomorphism $\Gamma \rightarrow \text{GL}_n(R)$ whose image is not virtually solvable.*
- (3) *There exists a field K , an integer $n > 0$ and group homomorphism $\Gamma \rightarrow \text{GL}_n(K)$ whose image is not virtually solvable.*
- (4) *There exists a finite index subgroup $\Gamma_0 < \Gamma$, a field K , a connected adjoint K -simple algebraic group \mathbf{G} defined over K and group homomorphism $\Gamma_0 \rightarrow \mathbf{G}(K)$ whose image is Zariski dense.*
- (5) *There exists a finite index subgroup $\Gamma_0 < \Gamma$, a local field k , a connected adjoint k -simple algebraic group \mathbf{G} defined over k and group homomorphism $\Gamma_0 \rightarrow \mathbf{G}(k)$ whose image is Zariski dense and unbounded (i.e. with non-compact closure).*

Proof. The implications (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) are all clear (recalling that the Zariski closure of a solvable group is solvable, see Corollary 1 in §2.4 of Chapter 1 in [10]). We will prove their converses.

(1) \Rightarrow (2) : We fix a commutative unital ring R , a finitely generated projective R -module M and a group homomorphism $\phi: \Gamma \rightarrow \text{Aut}_R(M)$ whose image is not virtually solvable. We let N be an R -module such that $M \oplus N \simeq R^n$ for some positive integer n and consider the map $\phi': \Gamma \rightarrow \text{Aut}(M \oplus N) \simeq \text{GL}_n(R)$ given by $\phi'(\gamma)(m, n) = (\phi(\gamma)m, n)$.

(2) \Rightarrow (3) : We fix a commutative unital ring R , a positive integer n and group homomorphism $\phi: \Gamma \rightarrow \text{GL}_n(R)$ whose image is not virtually solvable. Since Γ is finitely generated we assume as we may that R is finitely generated, hence Noetherian by Hilbert's Basis theorem (see [41, §7.10]). We let \mathfrak{n} denote the **nil radical** of R , i.e. the set of nilpotent elements of R . Since R is Noetherian, every ideal is finitely generated. Therefore there is a natural d such that $\mathfrak{n}^d = \{0\}$. By a theorem of Krull [41, Th. 7.1], the nil radical \mathfrak{n} is the intersection of all prime ideals. Moreover, a theorem of Noether ensures that the set of minimal prime ideals is finite (see Exercise 1.2 on p. 47 in [32]). In particular, denoting the minimal prime ideals by $\mathfrak{p}_1, \dots, \mathfrak{p}_m$, we infer that $\mathfrak{n} = \bigcap_{i=1}^m \mathfrak{p}_i$.

For every integer $e \geq 0$ we denote by $G(e)$ the kernel of the natural homomorphism $\text{GL}_n(R) \rightarrow \text{GL}_n(R/\mathfrak{n}^e)$. Observe that for all $e, f \geq 0$, we have $[G(e), G(f)] \leq G(e+f)$. Moreover we have $G(e) \leq G(f)$ if $e \geq f$. It follows that for $e \geq 1$, the quotient $G(1)/G(e)$ is nilpotent. In particular, the group $G(1) \simeq G(1)/G(d)$ is nilpotent. Let us denote by K_i the fraction field of R/\mathfrak{p}_i and by $\phi_i: \Gamma \rightarrow \text{GL}_n(K_i)$ the homomorphism obtained by composing ϕ with the natural homomorphism $\text{GL}_n(R) \rightarrow \text{GL}_n(R/\mathfrak{p}_i) \rightarrow \text{GL}_n(K_i)$. Since the image of Γ in $\text{GL}_n(R)$ is not virtually solvable and the kernel of $\text{GL}_n(R) \rightarrow \prod_{i=1}^m \text{GL}_n(K_i)$, which

equals $G(1)$, is nilpotent, we conclude that for some i , the image of Γ under ϕ_i is not virtually solvable.

(3) \Rightarrow (4) : We fix a field K , an integer $n > 0$ and group homomorphism $\phi : \Gamma \rightarrow \mathrm{GL}_n(K)$ whose image is not virtually solvable. We let \mathbf{H} be the Zariski closure of $\phi(\Gamma)$. By [10, AG, Theorem 14.4] \mathbf{H} is defined over K and $\phi(\Gamma) < \mathbf{H}(K)$. We denote by \mathbf{H}^0 the identity connected component in \mathbf{H} and observe that $\mathbf{H}^0(K) < \mathbf{H}(K)$ is of finite index. We set $\Gamma_0 = \phi^{-1}(\mathbf{H}^0(K))$ and deduce that $\Gamma_0 < \Gamma$ is of finite index. Note that the Zariski closure of $\phi(\Gamma_0)$ is \mathbf{H}^0 , as it is contained in the connected group \mathbf{H}^0 and it is of finite index in \mathbf{H} . We let \mathbf{R} be the solvable radical of \mathbf{H}^0 and set $\mathbf{L} = \mathbf{H}^0/\mathbf{R}$. Since $\phi(\Gamma)$ is not virtually solvable, the group $\phi(\Gamma_0)$ is not solvable, thus $\mathbf{R} \neq \mathbf{H}^0$ and \mathbf{L} is non-trivial. Upon replacing \mathbf{L} by its quotient modulo its center, which is finite, we may further assume that \mathbf{L} is centerless. Thus \mathbf{L} is a connected, adjoint, semisimple K -group and the image of $\phi(\Gamma_0)$ under $\mathbf{H}^0(K) \rightarrow \mathbf{L}(K)$ is Zariski dense in \mathbf{L} . The adjoint group \mathbf{L} is K -isomorphic to a direct product of connected, adjoint, K -simple K -groups. We let \mathbf{G} be one of these K -simple factors. We let $\phi_0 : \Gamma_0 \rightarrow \mathbf{G}(K)$ be the composition of ϕ with the natural map $\mathbf{H}^0(K) \rightarrow \mathbf{L}(K) \rightarrow \mathbf{G}(K)$ and conclude that $\phi_0(\Gamma_0)$ is Zariski dense in the connected adjoint K -simple K -algebraic group \mathbf{G} .

(4) \Rightarrow (5) : We fix a finite index subgroup $\Gamma_0 < \Gamma$, a field K , a connected adjoint K -simple K -algebraic group \mathbf{G} and a group homomorphism $\Gamma_0 \rightarrow \mathbf{G}(K)$ with a Zariski dense image. We choose an injective K -morphism $\mathbf{G} \rightarrow \mathrm{GL}_n$ for some n and let $\psi : \Gamma_0 \rightarrow \mathrm{GL}_n(K)$ be the composition of ϕ with the homomorphism $\mathbf{G}(K) \rightarrow \mathrm{GL}_n(K)$. We let I be set of matrix coefficients of $\psi(\Gamma_0)$. Let moreover R be the ring generated by I and F be its fraction field. Since $\phi(\Gamma_0)$ is Zariski dense in \mathbf{G} and is contained in $\mathbf{G}(F)$ we conclude by [10, AG, Theorem 14.4] that \mathbf{G} is defined over F . The group Γ is finitely generated, hence so is its finite index subgroup Γ_0 . It follows that R is a finitely generated domain. By [13, Lemma 2.1] there exists a local field k and an embedding $F \hookrightarrow k$ such that the image of I in k is unbounded. Since \mathbf{G} is defined over F we get that it is also defined over k , via this embedding. The composition of $\phi : \Gamma_0 \rightarrow \mathbf{G}(F)$ with the homomorphism $\mathbf{G}(F) \rightarrow \mathbf{G}(k)$ induced by the embedding $F \hookrightarrow k$ is again denoted by ϕ . We already know that $\phi(\Gamma_0)$ is Zariski dense in \mathbf{G} . Since the image of I is unbounded in k , we see that $\psi(\Gamma_0)$ is unbounded in $\mathrm{GL}_n(k)$, so that $\phi(\Gamma_0)$ is unbounded in $\mathbf{G}(k)$. \square

Remark A.2. The equivalence between (2) and (3) in Theorem A.1 remains if one replaces the phrases “virtually solvable” by either “solvable”, “finite” or “trivial”. The proof for the “solvable” case is the same, mutatis-mutandis. For the “finite” and “trivial” cases we need one extra ingredient: the argument above fails only if the image of ϕ is virtually nilpotent, but every finitely generated virtually nilpotent group has a faithful representation over \mathbf{Q} .

Theorem A.1 provides the following tool for proving non-linearity of groups.

Corollary A.3. *Let Γ be a finitely generated group such that every finite index subgroup has a finite abelization (e.g a group with property (T)). Assume that there is no homomorphism $\Gamma_0 \rightarrow \mathbf{G}(k)$ with an unbounded and Zariski dense image, where $\Gamma_0 < \Gamma$ is a finite index subgroup, k a local field and \mathbf{G} a connected adjoint k -simple k -algebraic group. Then every homomorphism $\Gamma \rightarrow \mathrm{GL}_n(R)$, where R is a commutative ring with 1, has a finite image.*

B The Hopf argument

The Hopf argument is a familiar ergodicity criterion. In our paper we need to use it in a form which is slightly more abstract than the standard one. This appendix is devoted to the formulation and proof of the Hopf argument in our needed form. In fact, as no further complexity is introduced, we allowed ourselves a little more general form than actually needed: for all applications we are aware of the group T considered below is abelian and the collection I consists of two elements.

As the formulation of the theorem below is not much simpler than its proof, we advice the reader to have in mind the following classical setting: M is a hyperbolic surface whose universal cover is identified with the hyperbolic plane \mathbf{H}^2 , $V = T^1(M)$ and $T = \mathbf{R}$ acts on it by the geodesic flow, $Z = T^1(\mathbf{H}^2)$ is endowed with the geodesic flow action of T and the deck transformation action of $\Lambda = \pi_1(M)$, $W_1 = W_2$ is the ideal boundary of \mathbf{H}^2 , $Y = W_1 \times W_2$ and $\psi: Z \rightarrow Y$ is given by the asymptotic hitting points of the geodesic flow.

Two sequences (a_n) and (b_n) in a compact space V are called **proximal** if for every neighborhood U of the diagonal in $V \times V$, we have $(a_n, b_n) \in U$ for all but finitely many n 's.

Theorem B.1 (The Hopf argument). *Let T be an amenable group and Λ be any group. Let Z be a locally compact topological space endowed with commuting actions of T and Λ . Assume both actions (separately) $\Lambda \curvearrowright Z$ and $T \curvearrowright Z$ are proper, let $V = Z/\Lambda$ and $Y = Z/T$ and let $\phi: Z \rightarrow V$ and $\psi: Z \rightarrow Y$ be the quotient maps. Assume V is compact. Note that there are natural actions $T \curvearrowright V$ and $\Lambda \curvearrowright Y$, the map ϕ is Λ -invariant and T -equivariant and the map ψ is T -invariant and Λ -equivariant. Let $\{W_i\}_{i \in I}$ be a collection of Λ -spaces and for each $i \in I$ let $\pi_i: Y \rightarrow W_i$ be a Λ -equivariant map, as described in the following diagram:*

$$\begin{array}{ccccc}
 & Z & & & \\
 \phi \swarrow & & \searrow \psi & & \\
 V & & & Y & \\
 & \pi_i \swarrow & & \searrow \pi_j & \\
 & W_i & \cdots & & W_j
 \end{array} \tag{7}$$

Assume for each $i \in I$ there exists a central element $t_i \in T$ such that for every $z, z' \in Z$ satisfying $\pi_i \psi(z) = \pi_i \psi(z')$ there exist $t, t' \in T$ such that the two sequences $(t_i^n t \phi(z))_{n \in \mathbb{N}}$ and $(t_i^n t' \phi(z'))_{n \in \mathbb{N}}$ are proximal in V .

Let m_Z be a $T \times \Lambda$ -invariant measure class on Z and denote its pushes to the various factors in the diagram by m_V, m_Y and m_{W_i} . Assume further:

- the measure class m_V contains a T -invariant probability measure.
- Any function in $\bigcap_{i \in I} \pi_i^*(L^\infty(W_i, m_{W_i}))^\Lambda$ is essentially constant.

Then the two actions $T \curvearrowright (V, m_V)$ and $\Lambda \curvearrowright (Y, m_Y)$ are ergodic.

Proof. First note that

$$L^\infty(V, m_V)^T \simeq L^\infty(Z, m_Z)^{T \times \Lambda} \simeq L^\infty(Y, m_Y)^\Lambda,$$

thus the ergodicity of $T \curvearrowright (V, m_V)$ is equivalent to that of $\Lambda \curvearrowright (Y, m_Y)$. We will prove the former.

Since the measure class m_V contains a T -invariant probability measure (which by an abuse of notation we keep calling m_V), it is enough to prove that every T -invariant function class in $L^2(V, m_V)$ is essentially constant, or equivalently, that the image of the orthogonal projection map $p_T: L^2(V, m_V) \rightarrow L^2(V, m_V)^T$ consists of the essentially constant functions. To this end, since $C(V)$ is dense in $L^2(V, m_V)$, it is enough to show that for every $f \in C(V)$, $p_T(f)$ is essentially constant. We now fix $f_0 \in C(V)$ and argue to show that $p_T(f_0)$ is essentially constant.

Fix a Følner sequence F_n in the amenable group T and define, for every bounded measurable function $f \in B(V)$ and for every $x \in V$,

$$\bar{f}(x) = \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{F_n} f(sx) ds$$

where the integration is with respect to a fixed left Haar measure on T and $|F_n|$ denotes the measure of F_n . We stress the fact that \bar{f} is an actual function and it is everywhere T -invariant, while on the contrary $p_T(f)$ is a class of functions. By the Mean Ergodic Theorem for amenable groups [37], \bar{f} is a function in the class $p_T(f)$.

Fix $i \in I$ and let $t_i \in T$ be a central element with the property described in the formulation of the theorem. Let $p_i: L^2(V, m_V) \rightarrow L^2(V, m_V)^{t_i}$ be the orthogonal projection onto the space of t_i -invariant functions and note that by the centrality of t_i , $p_T p_i = p_T$. Define for every $x \in V$

$$f_i(x) = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f_0(t_i^n x).$$

We obtain a bounded measurable function $f_i \in B(V)$. By Birkhoff Ergodic Theorem f_i is in the class $p_i(f_0)$.

Fix $z, z' \in Z$ satisfying $\pi_i \psi(z) = \pi_i \psi(z')$. By the defining property of t_i there exist $t, t' \in T$ such that the two sequences $(t_i^n t \phi(z))_{n \in \mathbb{N}}$ and $(t_i^n t' \phi(z'))_{n \in \mathbb{N}}$ are proximal in X . It follows that for every $s \in T$ the sequences $(t_i^n s t \phi(z))_{n \in \mathbb{N}}$ and $(t_i^n s t' \phi(z'))_{n \in \mathbb{N}}$ are proximal (we have used here the centrality of t_i again) and by the (uniform) continuity of f_0 , it is easy to see that $f_i(s t \phi(z)) = f_i(s t' \phi(z'))$. We deduce that $\bar{f}_i(t \phi(z)) = \bar{f}_i(t' \phi(z'))$. Since \bar{f}_i is T -invariant we get $\bar{f}_i(\phi(z)) = \bar{f}_i(\phi(z'))$. Note that \bar{f}_i is in the class of $p_T p_i(f_0) = p_T(f_0)$. In particular, its class is independent of $i \in I$.

Under the obvious isomorphism

$$B(V)^T \simeq B(Z)^{T \times \Lambda} \simeq B(Y)^\Lambda$$

\bar{f}_i corresponds to a function $g_i \in B(Y)^\Lambda$ which, by the above property, is a pull back of a function in $B(W_i)$. Since the maps ϕ and ψ are measure class preserving, we obtain that the class of g_i is independent of i . Denote this class by g and get $g \in \bigcap_{i \in I} \pi_i^*(L^\infty(W_i, m_{W_i}))^\Lambda$. Then by assumption g is essentially constant. We conclude that $p_T(f_0)$ is essentially constant, and the proof is complete. \square

C Galois lattices are non-linear

C.1 The automorphism group of a simple algebraic group

Given a locally compact group G , we denote by $\text{Aut}(G)$ the group of bi-continuous automorphisms of G , endowed with the **Braconnier topology** (also sometimes called the **Birkhoff**

topology). In general $\text{Aut}(G)$ need not be locally compact, but it will be so for the specific groups G that we shall consider. Given a locally compact field k , we also denote by $\text{Aut}(k)$ the group of bi-continuous automorphisms of k . If k is not isomorphic to \mathbf{C} , any automorphism of k is automatically continuous: this is clear if k is discrete or if $k \simeq \mathbf{R}$ since $\text{Aut}(\mathbf{R})$ is trivial, and follows otherwise from the uniqueness of the discrete valuation with respect to which k is complete. Moreover, if k is non-discrete, then $\text{Aut}(k)$ is infinite if and only if k has positive characteristic.

Let now \mathbf{G} be a k -simple adjoint algebraic group over a local field k , with k -rank ≥ 1 . The goal of this section is to show that if a lattice Γ in the locally compact group $\text{Aut}(\mathbf{G}(k))$ has infinite image in $\text{Out}(\mathbf{G}(k))$ and if \mathbf{G} has k -rank ≥ 2 , then every finite-dimensional linear representation of Γ has finite image. Before providing the formal statement of the theorem, we first recall some useful facts on semi-simple groups and their automorphisms.

The first point to keep in mind is that given a group \mathbf{G} over k as above, there is a semi-simple absolutely almost simple adjoint group \mathbf{H} defined over a finite separable extension k' of k , so that \mathbf{G} is the Weil restriction of \mathbf{H} from k' to k (see [11, §6.21(ii)]). In particular the locally compact groups $\mathbf{H}(k')$ and $\mathbf{G}(k)$ are isomorphic, so there is no loss of generality in assuming that \mathbf{G} is absolutely almost simple. After that reduction, a description of the automorphism group of $\mathbf{G}(k)$ is provided by the work of Borel and Tits [12]. We record the following points for the sake of future references.

Proposition C.1. *Let \mathbf{G} be an absolutely almost simple adjoint algebraic group over a local field k , with k -rank ≥ 1 . Let X denote the Bruhat–Tits building of $\mathbf{G}(k)$, and $\mathbf{G}(k)^+$ denote the subgroup generated by the unipotent radicals of k -parabolic subgroups. The following assertions hold.*

- (i) *We have $\text{Aut}(\mathbf{G}(k)) \cong \text{Aut}(\mathbf{G}(k)^+)$. Moreover, as an abstract group $\text{Aut}(\mathbf{G}(k))$ is an extension of the group $\text{Aut}(\mathbf{G})(k)$ of algebraic automorphisms by the group $\text{Aut}_{\mathbf{G}}(k)$ consisting of those automorphisms α of k such that the groups \mathbf{G} and ${}^{\alpha}\mathbf{G}$ are k -isomorphic.*
- (ii) *The normal subgroup $\text{Inn}(\mathbf{G}(k)) \cong \mathbf{G}(k)$ is of finite index in $\text{Aut}(\mathbf{G})(k)$. Moreover $\text{Aut}_{\mathbf{G}}(k)$ is an open subgroup of finite index in $\text{Aut}(k)$.*
- (iii) *If \mathbf{G} is k -split, then $\text{Aut}(\mathbf{G}(k)) \cong \mathbf{G}(k) \rtimes (A \times \text{Aut}(k))$, where A is the automorphism group of the Dynkin diagram of \mathbf{G} .*
- (iv) *$\text{Aut}(\mathbf{G}(k))$ is a compactly generated locally compact second countable group, the subgroup $\text{Inn}(\mathbf{G}(k)) \cong \mathbf{G}(k)$ is closed and cocompact. The intersection of all non-trivial closed normal subgroups of $\text{Aut}(\mathbf{G}(k))$ is non-trivial and coincides with the natural image of $\mathbf{G}(k)^+$ in $\text{Aut}(\mathbf{G}(k))$ (which is injective).*
- (v) *$\text{Aut}(\mathbf{G}(k))$ acts continuously, properly and faithfully on X . This action induces an isomorphism of topological groups $\text{Aut}(\mathbf{G}(k)) \cong N_{\text{Aut}(X)}(\mathbf{G}(k))$, where $\mathbf{G}(k)$ is viewed as a closed subgroup of $\text{Aut}(X)$.*
- (vi) *The map $\text{Aut}(\mathbf{G}(k)) \rightarrow \text{Aut}(X)$ is surjective if and only if the k -rank of \mathbf{G} is ≥ 2 .*

Proof. (i) We recall from [12, Prop. 6.14] that $\mathbf{G}(k)^+$ is a closed subgroup. Moreover every abstract automorphism of $\mathbf{G}(k)$ (resp. $\mathbf{G}(k)^+$) is continuous by [12, Prop. 9.8]. The identifications

$$\text{Aut}(\mathbf{G}(k)) \cong \text{Aut}(\mathbf{G}(k)^+) \quad \text{and} \quad \text{Aut}(\mathbf{G}(k)) / \text{Aut}(\mathbf{G})(k) \cong \text{Aut}_{\mathbf{G}}(k)$$

as abstract groups follow from [12, Cor. 8.13]. The conjugation action of $\mathbf{G}(k)$ induces an embedding $\mathbf{G}(k) \rightarrow \text{Aut}(\mathbf{G})(k)$ whose image in $\text{Aut}(\mathbf{G}(k))$ is $\text{Inn}(\mathbf{G}(k))$, and is thus normal.

(ii) The index of $\text{Inn}(\mathbf{G}(k)) \cong \mathbf{G}(k)$ in $\text{Aut}(\mathbf{G})(k)$ in view of [29, Th. 7.1.9]. To see that the index of $\text{Aut}_{\mathbf{G}}(k)$ in $\text{Aut}(k)$ is finite, we observe that, given any $\alpha \in \text{Aut}(k)$, the group ${}^{\alpha}\mathbf{G}$ obtained from \mathbf{G} by base change is just another k -form of the same ‘absolute’ group \mathbf{G} over the algebraic closure of k . We may then invoke the classification of semi-simple groups over local fields (see [66] and [14]) to establish that for a fixed k , the simple group \mathbf{G} has finitely many k -forms. It follows that ${}^{\alpha}\mathbf{G}$ takes only finitely many values when α runs over $\text{Aut}(k)$. In particular $\text{Aut}_{\mathbf{G}}(k)$ is of finite index, hence open, in $\text{Aut}(k)$ since $\text{Aut}(k)$ is a topologically finitely generated virtually pro- p group (see [19]).

(iii) If \mathbf{G} is k -split, then it is defined over the prime field of k , so that $\text{Aut}_{\mathbf{G}}(k) = \text{Aut}(k)$. The result then follows from (i) and [62, §16.3].

(iv) The natural embedding of $\mathbf{G}(k)$ into $\text{Aut}(\mathbf{G}(k))$ is continuous by definition of the Braconnier topology. A proof that $\text{Aut}(\mathbf{G}(k))$ is locally compact and second countable may be found in [24, Prop. 4.10] (formally, the latter statement applies to $\text{Aut}(\mathbf{G}(k)^+)$; the latter coincides with $\text{Aut}(\mathbf{G}(k))$ by (i)). Any continuous homomorphism of $\mathbf{G}(k)^+$ to a locally compact group has closed image (see [24, Prop. 2.1]). Since $\mathbf{G}(k)/\mathbf{G}(k)^+$ is compact by [12, Prop. 6.14], the same statement holds for $\mathbf{G}(k)$. Hence $\text{Inn}(\mathbf{G}(k))$ is closed in $\text{Aut}(\mathbf{G}(k))$. Since $\mathbf{G}(k)^+$ is compactly generated (see e.g. [24, Th. 2.2]), so is $\mathbf{G}(k)$, and all remaining assertions of (iii) will follow once we establish that $\text{Aut}(\mathbf{G}(k))/\text{Inn}(\mathbf{G}(k))$ is compact. We postpone the proof of this to the next paragraph.

(v) Let p be the residue characteristic of k . The chamber-stabilizers in X are precisely the normalizers of the maximal pro- p subgroups of $\mathbf{G}(k)$ (see the discussion in [66, §3.7]). Moreover two distinct chambers are adjacent if and only if the index of the intersection of their stabilizers is minimal > 1 . Those observations imply that $\text{Aut}(\mathbf{G}(k))$ naturally acts by automorphisms on the building X . By definition of the Braconnier topology, the normalizer of any compact open subgroup of $\mathbf{G}(k)$ is open in $\text{Aut}(\mathbf{G}(k))$. This implies that every vertex-stabilizer for the $\text{Aut}(\mathbf{G}(k))$ -action on X is open. Thus the latter action is continuous. To see that it is faithful, observe that the kernel K of this action normalizes every parahoric subgroup, and also every parabolic subgroup of $\mathbf{G}(k)$. In particular it also normalizes the unipotent radical U of every minimal parabolic P . Since the latter unipotent radical U acts freely and transitively on the parabolic subgroups opposite P , it follows that K acts trivially on U . This holds for any P and U , so that K acts trivially on $\mathbf{G}(k)^+$. Therefore K is trivial by (i). Thus we have obtained a continuous injective homomorphism $\text{Aut}(\mathbf{G}(k)) \rightarrow \text{Aut}(X)$. It takes values in the normalizer $N_{\text{Aut}(X)}(\mathbf{G}(k))$ (where $\mathbf{G}(k)$ has been identified with its image in $\text{Aut}(X)$). The conjugation action of $N_{\text{Aut}(X)}(\mathbf{G}(k))$ on $\mathbf{G}(k)$ also yields a continuous map $N_{\text{Aut}(X)}(\mathbf{G}(k)) \rightarrow \text{Aut}(\mathbf{G}(k))$ which is injective since any automorphism of X commuting with $\mathbf{G}(k)$ has constant displacement function (because $\mathbf{G}(k)$ is transitive on the chambers), and must thus be trivial. It follows that the map $\text{Aut}(\mathbf{G}(k)) \rightarrow N_{\text{Aut}(X)}(\mathbf{G}(k))$ is bijective. Since $\mathbf{G}(k)$ is closed in $\text{Aut}(X)$, it follows that $N_{\text{Aut}(X)}(\mathbf{G}(k))$ is closed. Therefore the continuous isomorphism $\text{Aut}(\mathbf{G}(k)) \rightarrow N_{\text{Aut}(X)}(\mathbf{G}(k))$ is an isomorphism of topological groups by Baire, since $\text{Aut}(\mathbf{G}(k))$ is second countable. This implies that the $\text{Aut}(\mathbf{G}(k))$ -action on X is proper. Since $\mathbf{G}(k)$ acts cocompactly on X , it is a cocompact subgroup of $\text{Aut}(X)$, and hence of $N_{\text{Aut}(X)}(\mathbf{G}(k))$. The compactness of $\text{Aut}(\mathbf{G}(k))/\text{Inn}(\mathbf{G}(k)) \cong N_{\text{Aut}(X)}(\mathbf{G}(k))/\mathbf{G}(k)$ follows.

(vi) If G has k -rank one, then X is a semi-regular tree. The group $\text{Aut}(X)$ is then much

bigger than $\text{Aut}(\mathbf{G}(k))$. Indeed the group $\text{Aut}(X)^+$ of type-preserving automorphisms of X is simple ([64]) and of index ≤ 2 in $\text{Aut}(X)$. If the map $\text{Aut}(\mathbf{G}(k)) \rightarrow \text{Aut}(X)$ were surjective, it would be an isomorphism of topological groups (by Baire) so that it would induce an isomorphism of $\mathbf{G}(k)^+$ onto $\text{Aut}(X)^+$ by (iii). It is easy to see that a vertex stabilizer in $\text{Aut}(X)^+$ contains finite soluble groups of arbitrary large derived length, so that $\text{Aut}(X)^+$ is non-linear, a contradiction.

If G has k -rank ≥ 2 , then $\text{Aut}(X)$ coincides with the full automorphism group of the spherical building at infinity Δ (see [73, Th. 27.6]). The fact every element of $\text{Aut}(\Delta)$ comes from an element of $\text{Aut}(\mathbf{G}(k))$ is due to Tits (see [65, Cor. 5.9]). \square

Remark C.2. By Proposition C.1(ii), the short exact sequence

$$1 \rightarrow \text{Aut}(\mathbf{G})(k) \rightarrow \text{Aut}(\mathbf{G}(k)) \rightarrow \text{Aut}_{\mathbf{G}}(k) \rightarrow 1$$

splits as soon as \mathbf{G} is k -split. For non-split groups, the splitting of the short exact sequence may fail: explicit examples have been found by T. Stulemeijer [63].

C.2 Galois lattices

The main result of this section is the following.

Theorem C.3. *Let \mathbf{G} be a k -simple algebraic group defined over a local field k , of k -rank ≥ 2 . Let Γ be a lattice in $\text{Aut}(\mathbf{G}(k))$.*

If Γ has infinite image in $\text{Out}(\mathbf{G}(k))$, then for any commutative unital ring R and any $n \geq 1$, any homomorphism $\Gamma \rightarrow \text{GL}_n(R)$ has finite image.

Since $\text{Out}(\mathbf{G}(k))$ is virtually a subgroup of the Galois group $\text{Aut}(k)$ by Proposition C.1, a lattice as in the theorem with infinite image in $\text{Out}(\mathbf{G}(k))$ will be called a **Galois lattice**. The existence of irreducible Galois lattices in higher rank is an open problem (which is explicitly mentioned as Problem 19 in the problem list [34]). In the rank one case, cocompact Galois lattices indeed exist, as shown by the following.

Proposition C.4. *Let k be a local field of characteristic $p > 0$, with residue field of order q . Given any q -generated residually p -group Λ , there is a uniform lattice in $G = \text{SL}_2(k) \rtimes \text{Aut}(k)$ whose projection to $\text{Aut}(k) \cong G/\text{SL}_2(k)$ is isomorphic to Λ .*

Proof. Let T be the Bruhat–Tits tree of G , which is regular of degree $q + 1$. We denote by $\text{Aut}(T)^+$ the index 2 subgroup of $\text{Aut}(T)$ consisting of the type-preserving automorphisms. Remark that G acts by type-preserving automorphisms on T .

Fix a base edge $\{v, w\} \in E(T)$. Let $e_1, \dots, e_q \in E(T)$ (resp. $f_1, \dots, f_q \in E(T)$) denote the edges different from $\{v, w\}$ and containing v (resp. w). We shall use the following.

Claim. *Let $g_1, \dots, g_q \in \text{Aut}(T)^+$ be such that $g_i(e_i) = f_i$. Then $\Gamma = \langle g_1, \dots, g_q \rangle$ is a discrete free subgroup of $\text{Aut}(T)^+$ acting cocompactly on T . In particular Γ is a uniform lattice in $\text{Aut}(T)$.*

Since g_i is type-preserving and maps the edge e_i to an edge at distance 1, it must be a translation. The fact that Γ is free and discrete then follows from a ping-pong argument. For the cocompactness, consider an arbitrary edge $e \in E(T)$. If e does not contain v or w , then the geodesic joining e to the base edge $\{v, w\}$ contains e_i or f_i for some i , and the edge $g_i(e)$

or $g_i^{-1}(e)$ is strictly closer to $\{v, w\}$ than e . This proves by induction that the set of edges containing v or w contains a fundamental domain for the Γ -action. The claim stands proven.

We apply the claim as follows. Since $\mathrm{SL}_2(k)$ is edge-transitive on T , we may find $\gamma_i \in \mathrm{SL}_2(k)$ mapping e_i to f_i . We next recall that $\mathrm{Aut}(k)$ is compact and acts continuously on T (by Proposition C.1), so that the subgroup of $\mathrm{Aut}(k)$ fixing all edges containing v or w is open of finite index. We next invoke the fact, due to Rachel Camina [18], that $\mathrm{Aut}(k)$, and hence also every open subgroup of $\mathrm{Aut}(k)$, contains a copy of every countably based pro- p group. We may thus choose $s_1, \dots, s_q \in \mathrm{Aut}(k)$ so that s_i fixes every edge of T containing v or w and that $\langle s_1, \dots, s_q \rangle$ is isomorphic to Λ . Finally we put $g_i = \gamma_i s_i$. The claim ensures that $\Gamma = \langle g_1, \dots, g_q \rangle$ is a uniform lattice in $G = \mathrm{SL}_2(k) \rtimes \mathrm{Aut}(k)$. By construction, its image in $\mathrm{Aut}(k)$ is $\langle s_1, \dots, s_q \rangle \cong \Lambda$. (A related argument which also provides a lifting of free groups to Galois groups appears in [65, §11.14].) \square

By Proposition C.1, we deduce the following statement on lattices in Bruhat–Tits buildings.

Corollary C.5. *Let X be the Bruhat–Tits building associated with a k -simple algebraic group \mathbf{G} defined over a local field k , of k -rank ≥ 2 .*

A lattice $\Gamma \leq \mathrm{Aut}(X)$ admits a finite-dimensional linear representation with infinite image if and only if Γ is virtually contained (necessarily as an arithmetic subgroup) in the image of $\mathbf{G}(k)$ in $\mathrm{Aut}(X)$.

Proof. The sufficiency of the condition is clear. For the reverse implication, notice first by [12, Cor. 6.5 and Rem. 6.15] that the quotient $\mathbf{G}(k)^+/Z$ is isomorphic to $\mathbf{G}_{\mathrm{ad}}(k)^+$, where $\mathbf{G}_{\mathrm{ad}} = \mathbf{G}/Z(\mathbf{G})$ is adjoint. Hence, by Proposition C.1 we have $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbf{G}(k)^+/Z) \cong \mathrm{Aut}(\mathbf{G}_{\mathrm{ad}}(k))$. This allows us to invoke Theorem C.3. The desired conclusion follows. \square

C.3 Proof of non-linearity

This section is devoted to the proof of Theorem C.3.

The following notation will be used throughout. The characteristic subgroup of $\mathbf{G}(k)$ generated by the unipotent radicals of k -parabolic subgroups is denoted by $\mathbf{G}(k)^+$. The image of $\mathbf{G}(k)^+$ in $\mathrm{Aut}(\mathbf{G}(k))$, which is a closed normal subgroup, is denoted by G^+ . We also denote by X the Bruhat–Tits building of $\mathbf{G}(k)$. We may identify $\mathrm{Aut}(\mathbf{G}(k))$ with $\mathrm{Aut}(X)$ in view of Proposition C.1.

Upon replacing Γ by a finite index subgroup, we may and shall assume that the Γ -action on X is type-preserving and that $\Gamma/\Gamma \cap \mathrm{Inn}(\mathbf{G}(k))$ maps injectively to $\mathrm{Aut}_{\mathbf{G}}(k)$ under the map $\mathrm{Aut}(\mathbf{G}(k)) \rightarrow \mathrm{Aut}_{\mathbf{G}}(k)$ afforded by Proposition C.1 (see (i) and (ii) from the latter proposition). Now we set

$$G = \overline{G^+ \Gamma}.$$

The quotient G/G^+ is compact by Proposition C.1.

We assume that Γ has a linear representation over a commutative unital ring with infinite image. Our goal is thus to show that G/G^+ is finite.

Let $A \subset X$ be an apartment and $\mathrm{Fix}_G(A)$ be the pointwise stabilizer of A in G . Thus $\mathrm{Fix}_G(A)$ is compact, and we have

$$G = G^+ \mathrm{Fix}_G(A)$$

since G^+ is transitive on the set of apartments of X . Fix a chamber C at infinity of A and let π_1, \dots, π_m be the panels of C . Recall that the stabilizer $\text{Stab}_G(\pi_j)$ acts on a locally finite tree \mathcal{T}_j , which is the panel tree associated with π_j (see [73, 11.18]). The set of ends of \mathcal{T}_j is in canonical one-to-one correspondence with the set of chambers of Δ incident with the panel π_j (the corresponding statement in the case of \tilde{A}_2 -buildings was recalled in Proposition 2.15).

Let π'_j be the panel at infinity of A which is opposite π_j and set

$$L_j = \text{Stab}_G(\pi_j, \pi'_j) \quad \text{and} \quad T_j = \text{Ker } \varphi_j,$$

where

$$\varphi_j: L_j \rightarrow \text{Aut}(\mathcal{T}_j)$$

is the natural action whose existence has just been recalled. Let also C' be the chamber at infinity of A which is opposite C and set

$$T = \text{Stab}_G(C, C').$$

The following result relies crucially on the hypothesis that the k -rank of \mathbf{G} is ≥ 2 .

Lemma C.6. *$T_j \cap G^+$ is not compact.*

Proof. Indeed $T_j \cap G^+$ contains a split torus of codimension 1 in G^+ , and is thus non-compact by hypothesis on the rank. \square

Lemma C.7. *The Γ -actions on G/T_j and on G/T are ergodic.*

Proof. By Moore's inversion trick [75, Cor. 2.2.3], it suffices to show that T_j acts ergodically on H/Γ , hence that $T_j \cap G^+$ acts ergodically on G/Γ . In view of Lemma C.6, the Howe–Moore property for G^+ (see [40]) reduces the problem to showing that G^+ acts ergodically on G/Γ . Now we apply Moore's inversion again, and see that the conclusion will follow if we show that Γ is ergodic on G/G^+ . By definition Γ has dense image in the compact quotient G/G^+ ; the desired ergodicity is now a consequence of the Lebesgue Density Theorem (see [1, Lem. 7] for a detailed proof).

Since $T_j \leq T$, the space G/T is a Γ -quotient of G/T_j , and is thus ergodic as well. \square

Lemma C.8. *The group $M_j = L_j/T_j$ maps continuously, injectively onto a closed subgroup of $\text{Aut}(\mathcal{T}_j)$ which is 2-transitive on $\partial\mathcal{T}_j$. In particular M_j has a smallest non-trivial closed normal subgroup M_j^+ which is cocompact, compactly generated, topologically simple and non-discrete.*

Moreover M_j^+ is isomorphic to the quotient of the derived group of the rank one Levi factor $L_j \cap S = \text{Stab}_S(\pi_j, \pi'_j)$ by its center.

Proof. The fact that the homomorphism $L_j/T_j \rightarrow \text{Aut}(\mathcal{T}_j)$ is continuous and injective is clear from the definition. We have $G = G^+ \text{Fix}_G(A)$. Moreover $\text{Fix}_G(A) \leq \text{Stab}_G(\pi_j, \pi'_j) = L_j$. It follows that

$$L_j = (L_j \cap G^+) \text{Fix}_G(A).$$

Since $\text{Fix}_G(A)$ is compact, the closedness of the image of L_j follows from that of $L_j \cap G^+$. Notice that $L_j \cap G^+ = \text{Stab}_{G^+}(\pi_j, \pi'_j)$ is a Levi factor of the parabolic subgroup $\text{Stab}_{G^+}(\pi_j)$. By construction, the semi-simple part of that Levi factor is of relative rank one, and \mathcal{T}_j is isomorphic to its Bruhat–Tits tree. The assertions of closedness and of boundary-2-transitivity follow, recalling that a semi-simple group over a local field acts properly on its Bruhat–Tits building (see Proposition C.1(v)). The remaining assertions follows from Theorem 3.7. \square

Lemma C.9. Γ has Kazhdan's property (T).

Proof. Indeed, the group $\mathbf{G}(k)$ has property (T) by [8, Th. 1.6.1]. Now, combining [8, Th. 1.7.1] and Proposition C.1(iv)), we see that this property is inherited by the locally compact group $\text{Aut}(X)$, and then by the lattice Γ . \square

In the following step, we invoke again Bader–Furman's theory outlined in Section 8.1.

Lemma C.10. *There exists $j \in \{1, \dots, m\}$ such that group $M_j = L_j/T_j$ has a continuous faithful linear representation over a local field.*

Proof. By Lemma C.9, we may apply Corollary A.3. This ensures that the given representation of Γ may be assumed to have an unbounded and Zariski-dense image in a simple algebraic group \mathbf{J} over a local field K . We may then apply the Bader–Furman theory with ergodic Γ space $Y = H/T_j$. The group $N_G(T_j)/T_j$ has a natural action on Y which commutes with the Γ -action. In particular, so does $M_j = L_j/T_j$ since $T_j \leq L_j \leq N_G(T_j)$. It then follows from Proposition 8.5 that M_j has a continuous representation over the same local field K . In order to prove that this representation is faithful, it suffices by Lemma C.8 to show that its restriction to M_j^+ is non-trivial.

To this end, we proceed as in the proof of Theorem 1.5.

In a first step, we observe that the gate of the ergodic Γ -space $Y_j = G/T_j$ from Lemma C.7 is non-trivial. Indeed, the Γ -action on G/P , where $P = \text{Stab}_G(C)$, is amenable because P is amenable (see [75, Prop. 4.3.2]). Moreover, it is doubly ergodic by Lemma C.7 since $G/P \times G/P$ is isomorphic to G/T as a Γ -space. It then follows from [4, Theorem 6.1] and [36, Theorem 1.1] that the gate of the Γ -space G/P is non-trivial. So is thus the gate \mathbf{J}/\mathbf{H}_Z of its extensions $Z = G/T$, and the gate \mathbf{J}/\mathbf{H}_j of $Y_j = G/T_j$.

In a second step, we observe that the longest element w_0 of the Weyl group of $N_G(T)/T$ acts as the flip on $G/P \times G/P$, and therefore has a non-trivial image under the representation $N_G(T)/T \rightarrow N_{\mathbf{J}}(\mathbf{H}_Z)/\mathbf{H}_Z(K)$ afforded by Proposition 8.5. This follows from the same argument as in Step 2 of the proof of Theorem 1.5.

We now conclude the proof of the lemma as follows. For each $j \in \{1, \dots, m\}$, choose an element $\tau_j \in L_j^+$ (where L_j^+ denotes the preimage of M_j^+ in L_j) acting as a reflection of A fixing the panel π_j . Then the natural images of $\{\tau_1, \dots, \tau_m\}$ in the Weyl group $N_G(T)/T$ is a generating set of that Weyl group. Therefore, if the image of τ_j under the continuous representation $N_G(T_j)/T_j \rightarrow N_{\mathbf{J}}(\mathbf{H}_j)/\mathbf{H}_j(K)$ afforded by Proposition 8.5 were trivial for all j , then so would be the image of w_0 under the representation $N_G(T)/T \rightarrow N_{\mathbf{J}}(\mathbf{H}_Z)/\mathbf{H}_Z(K)$ as a consequence of Proposition 8.6. This would contradict the conclusion of the second step. Therefore we find some $j \in \{1, \dots, m\}$ such that the image of τ_j under the representation $N_G(T_j)/T_j \rightarrow N_{\mathbf{J}}(\mathbf{H}_j)/\mathbf{H}_j(K)$ is non-trivial. In particular the image of M_j^+ is non-trivial. Therefore the restriction of that representation to M_j is faithful in view of Lemma C.8. \square

Lemma C.11. *The quotient M_j/M_j^+ is virtually abelian, where $j \in \{1, \dots, m\}$ is the element afforded by Lemma C.10.*

Proof. Follows from Proposition 3.8(iv), whose hypotheses are satisfied in view of Lemmas C.8 and C.10. \square

Lemma C.12. *If $h \in \text{Fix}_G(A)$ has non-trivial image in the Galois group $\text{Aut}_{\mathbf{G}}(k)$ under the map $\text{Aut}(\mathbf{G}(k)) \rightarrow \text{Aut}_{\mathbf{G}}(k)$, then for any $i \in \{1, \dots, m\}$ we have $\varphi_i(h) \notin M_i^+$.*

Proof. This follows from how the proof of the Borel–Tits theorem works in [12, §8]. Here are more details. Since $h \in H_A$, it acts trivially on the apartment A and, hence, it normalizes a maximal k -split torus \mathbf{S} of $\mathbf{G}(k)$ and as well as each of the root groups associated with the k -roots of \mathbf{S} . In §§8.2 and 8.3 of [12], the unique field automorphism involved in the automorphism h of $\mathbf{G}(k)$ is constructed by considering the induced action of h on a specific 3-dimensional k -split reductive subgroup L of a rank one Levi subgroup Q of $\mathbf{G}(k)$ associated to a k -root of \mathbf{S} . Those arguments in [12] imply that the h -action on L involves a field automorphism if and only if the h -action on $\mathbf{G}(k)$ does. For the same reason (viewing now h as an automorphism of the Levi factor Q), we see that the h -action on L involves a field automorphism if and only if the h -action on Q does. It is then showed in §§8.2 of [12] that the h -action on the rank one Levi factor associated to all k -roots of \mathbf{S} induce field automorphisms that differ from one another by a power of the Frobenius automorphism. Since local fields of positive characteristic are not perfect, the Frobenius map is not a field automorphism, so in our setting the h -action on each rank one Levi factor involves the same field automorphism.

Recalling from Lemma C.8 that M_i^+ is the quotient of a rank one Levi factor of G^+ divided by its center, we deduce from the discussion above that if $\varphi_i(h) \in M_i^+$, then h does not involve any non-trivial field automorphism, so that its image in $\text{Aut}_{\mathbf{G}}(k)$ is trivial. \square

Lemma C.13. *Let $\tilde{G} = \text{Inn}(\mathbf{G}(k)) \cap G$. Then the quotient $\text{Fix}_G(A)/\tilde{G} \cap \text{Fix}_G(A)$ is virtually abelian.*

Proof. Let $h, h' \in \text{Fix}_G(A)$ such that $h^{-1}h' \notin \tilde{G}$. Recall from the beginning of the proof that $\Gamma/\Gamma \cap \tilde{G}$ maps injectively to the Galois group. Therefore, we see that h and h' act as distinct field automorphisms on the group G^+ . Thus they have distinct images in M_i/M_i^+ for all i by Lemma C.12. This means that the kernel of the composed homomorphism $\text{Fix}_G(A) \xrightarrow{\varphi_i} M_i \rightarrow M_i/M_i^+$ is contained in \tilde{G} . In particular $\text{Fix}_G(A)/\tilde{G} \cap \text{Fix}_G(A)$ maps injectively to a quotient of M_i/M_i^+ . Applying that assertion to the index $i = j$ afforded by Lemma C.11, we obtain the required conclusion. \square

Lemma C.14. *The quotient G/G^+ is finite.*

Proof. The group \tilde{G}/G^+ is virtually abelian by [12, Prop. 6.14]. So is thus $\tilde{G} \cap \text{Fix}_G(A)/G^+ \cap \text{Fix}_G(A)$. In view of Lemma C.13, we infer that $\text{Fix}_G(A)/G^+ \cap \text{Fix}_G(A)$ is virtually metabelian. On the other hand, we have $G/G^+ = G^+ \text{Fix}_G(A)/G^+ \cong \text{Fix}_G(A)/G^+ \cap \text{Fix}_G(A)$, so that G/G^+ is virtually metabelian. Since Γ has property (T) by Lemma C.9 and dense image in G/G^+ by construction, we conclude that G/G^+ is indeed finite. \square

Lemma C.14 concludes the proof of Theorem C.3.

D Lattices in exotic \tilde{A}_2 -buildings of arbitrary large order

The goal of this section is to provide an explicit construction of an infinite family of lattices in exotic \tilde{A}_2 -buildings of arbitrarily large order. This relies on a construction, originally due to M. Ronan [58] and W. Kantor [43], relating panel-regular \tilde{A}_2 -lattices with finite difference sets. That construction was recently concretized by means of explicit group presentations by J. Essert [33], see Theorem D.3 below. Using those presentations together with Theorem 1.1, we shall see that the exoticity of a panel-regular lattice in a building of order q_0 typically implies the exoticity of many such buildings whose orders are suitable powers of q_0 . A concrete

illustration is described in Corollary D.10. Actually, our discussion will show that many of the \tilde{A}_2 -buildings admitting a panel-regular lattice are exotic. As mentioned in the introduction, this provides a partial answer to W. Kantor's question mentioned in the remarks following Corollary C.3.2 in [43].

D.1 Difference sets

A **difference set** in a group A is a subset $D \subset A$ such that any non-trivial element $a \in A$ can be written in a unique way as a product $a = d_1 d_2^{-1}$ with $d_1, d_2 \in D$. If A is finite of order n , the existence of a difference set D implies that $n = q^2 + q + 1$, where $q = |D| - 1$. A difference set $D \subset A$ naturally yields a projective plane $\mathcal{P}(A, D)$ whose points are the elements of A and whose lines are the translates of D in A . Conversely, given a finite projective plane \mathcal{P} and a group A acting freely and transitively on the points of \mathcal{P} , one obtains a difference set D in A by virtue of the following classical fact.

Lemma D.1. *Let \mathcal{P} be a finite projective plane and $A \leq \text{Aut}(\mathcal{P})$ a group acting freely and transitively on the points of \mathcal{P} . Then A acts freely and transitively on the lines of \mathcal{P} . Moreover, for any point p and any line L , the set $D = \{a \in A \mid a(p) \in L\}$ is a difference set in A .*

Moreover, the Desarguesian plane $\mathcal{P} = \mathcal{P}(2, \mathbf{F}_q)$ of order q admits a cyclic group A acting freely and transitively: namely, the canonical image of the multiplicative group $\mathbf{F}_{q^3}^\times$ in the group $\text{PGL}_3(\mathbf{F}_q)$, which is isomorphic to $\mathbf{F}_{q^3}^\times / \mathbf{F}_q^\times$.

Proof. See Assertion 4.2.7 and the discussion on pp. 105–106 in [31]. \square

The cyclic group from Lemma D.1 was first pointed out by J. Singer [61]; it is called a **Singer group**.

We shall be interested in pairs of finite groups A_0, A and difference sets $D_0 \subset A_0$, $D \subset A$ together with an injective homomorphism $A_0 \rightarrow A$ mapping D_0 into D . The following shows that such configurations may be constructed using Singer groups.

Lemma D.2. *Let p be a prime, let q_0 be a positive power of p and $q = q_0^e$ a positive power of q_0 . Assume that $q_0 \not\equiv 1 \pmod{3}$ and that $e \not\equiv 0 \pmod{3}$. Then $n_0 = q_0^2 + q_0 + 1$ divides $n = q^2 + q + 1$, and there exist difference sets $D_0 \subset \mathbf{Z}/n_0\mathbf{Z}$ and $D \subset \mathbf{Z}/n\mathbf{Z}$ such that $\{0, 1\} \subset D_0$ and $\{dn/n_0 \mid d \in D_0\} \subseteq D$.*

If $p = q_0 = 2$, then we may further arrange that $D_0 = \{0, 1, 3\}$.

Proof. Write $e = 3f + i$, with $i = 1$ or 2 . Since $q_0^{3f} - 1$ is a multiple of n_0 , we have $\gcd(n_0, q - 1) = \gcd(n_0, q_0^{3f}(q_0^i - 1))$. The hypotheses imply that $\gcd(n_0, q_0^{3f}) = 1 = \gcd(n_0, q_0 - 1) = \gcd(n_0, q_0 + 1)$, so that $\gcd(n_0, q - 1) = 1$. This implies that the canonical embedding of cyclic groups $\mathbf{F}_{q_0^3}^\times \rightarrow \mathbf{F}_{q^3}^\times$ descends to an injective homomorphism $\varphi: \mathbf{F}_{q_0^3}^\times / \mathbf{F}_{q_0}^\times \rightarrow \mathbf{F}_{q^3}^\times / \mathbf{F}_q^\times$. In particular n_0 divides n .

The inclusion $\mathbf{F}_{q_0} \subset \mathbf{F}_q$ yields an inclusion of projective planes $\iota: \mathcal{P}(2, \mathbf{F}_{q_0}) \rightarrow \mathcal{P}(2, \mathbf{F}_q)$. Identifying the Singer groups $A = \mathbf{F}_{q^3}^\times / \mathbf{F}_q^\times$ and $B = \mathbf{F}_{q_0^3}^\times / \mathbf{F}_{q_0}^\times$ with their natural images in $\text{PGL}_3(\mathbf{F}_q)$ and $\text{PGL}_3(\mathbf{F}_{q_0})$, we infer that the inclusion $\iota: \mathcal{P}(2, \mathbf{F}_{q_0}) \rightarrow \mathcal{P}(2, \mathbf{F}_q)$ is equivariant with respect to φ . Let a be a generator of A and b a generator of B such that $\varphi(b) = a^{n/n_0}$.

Let now x_0 a point of $\mathcal{P}(2, \mathbf{F}_{q_0})$ and L_0 be a line of $\mathcal{P}(2, \mathbf{F}_{q_0})$ containing x_0 and $b(x_0)$. Then

$$D_0 = \{d \in \mathbf{Z}/n_0\mathbf{Z} \mid b^d(x_0) \in L_0\}$$

is a difference set in $\mathbf{Z}/n\mathbf{Z}$ containing 0 and 1 by Lemma D.1. Let also $x = \iota(x_0)$ and L be a line of $\mathcal{P}(2, \mathbf{F}_q)$ containing $\iota(L_0)$. Then

$$D = \{d \in \mathbf{Z}/n\mathbf{Z} \mid a^d(x) \in L\}$$

is a difference set in $\mathbf{Z}/n\mathbf{Z}$ by Lemma D.1. Since $\varphi(b) = a^{n/n_0}$, we have $\{dn/n_0 \mid d \in D_0\} \subseteq D$ as required.

Assume finally that $p = q_0 = 2$, so that $n_0 = 7$. Observe that the only difference sets in $\mathbf{Z}/7\mathbf{Z}$ containing 0 and 1 are $\{0, 1, 3\}$ and $\{0, 1, 5\}$. Since $q^2 + q + 1$ is not divisible by 5, the element a^5 is also a generator of A . Therefore, in case we obtain $D_0 = \{0, 1, 5\}$, we redo the construction with a replaced by a^5 and b by b^5 . This yields $D_0 = \{0, 1, 3\}$. \square

D.2 Panel-regular lattices and their presentations

A type-preserving automorphism group of an \tilde{A}_2 -building X acting freely and transitively on the set of panels of each type is called a **panel-regular lattice**. M. Ronan [58, Prop. 3.2 and Theorem 3.3] and W. Kantor [43, Theorem C.3.1 and Cor. C.3.2] described a tight relation between difference sets in groups of order $n = q^2 + q + 1$ and panel-regular lattices in \tilde{A}_2 -buildings of order q . That relation was made more explicit by J. Essert [33], who characterized panel-regular lattices by means of an explicit and elegant presentation. The following result is due to him.

Theorem D.3 (J. Essert). *Let $q > 1$ be an integer, $n = q^2 + q + 1$ and D be a difference set containing 0 in the cyclic group $\mathbf{Z}/n\mathbf{Z}$. For all permutations $\pi_1, \pi_2 \in \text{Sym}(D)$ fixing 0, the group*

$$\Gamma = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_0^n, \sigma_1^n, \sigma_2^n, \sigma_0^d \sigma_1^{\pi_1(d)} \sigma_2^{\pi_2(d)} \forall d \in D \rangle$$

is a panel-regular lattice in an \tilde{A}_2 -building X of order q . Moreover there is a chamber with vertices v_0, v_1, v_2 such that the stabilizer Γ_{v_i} coincides with the cyclic group $\langle \sigma_i \rangle$ for $i = 0, 1, 2$.

Proof. See [33, Theorem 5.8]. The last assertion on the geometric action of the cyclic groups $\langle \sigma_i \rangle$ follows from [33, Corollary 5.7]. \square

We shall need the following basic fact on panel-regular lattices.

Lemma D.4. *Let X be an \tilde{A}_2 -building of order q and $\Gamma \leq \text{Aut}(X)$ be a panel-regular lattice. Let v, v' be vertices of X that have the same type and are distance 2 apart in the 1-skeleton. Then the set $\{\gamma \in \Gamma \mid \gamma(v) = v'\}$ contains exactly 2 elements of finite order, and $q^2 + q - 1$ elements of infinite order.*

In particular, if Γ is the panel-regular lattice given by the presentation from Theorem D.3, then for any non-zero $d \in D$ and any $e \in \mathbf{Z}/n\mathbf{Z}$, the element $\gamma = \sigma_0^d \sigma_1^e$ is of finite order if and only if $e \in \{0, \pi_1(d)\}$.

Proof. Since the Γ -action on X is type-preserving, an element $\gamma \in \Gamma$ of finite order must fix a vertex x . Since the $\langle \gamma \rangle$ -action on the link of x is free on panels of each type, we infer that for each $y \in X$ different from x , we have the bound $\angle_x(y, \gamma y) \geq 2\pi/3$ on the Alexandrov angle (in the sense of CAT(0) geometry) as soon as γ is non-trivial. This implies that if γ maps a vertex v to a vertex v' at distance 2 in the 1-skeleton, the only possible fixed vertex of γ is one of the two vertices, say v_1 and v_2 , that have distance 1 to both v and v' . An element γ fixing v_i and mapping v to v' maps the edge $[v_i, v]$ to $[v_i, v']$. Since edges correspond to

panels, the panel-regularity of Γ implies that there is exactly one element of Γ fixing v_i and mapping v to v' .

Let us now consider $\Gamma = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_0^n, \sigma_1^n, \sigma_2^n, \sigma_0^d \sigma_1^{\pi_1(d)} \sigma_2^{\pi_2(d)} \forall d \in D \rangle$ as in Theorem D.3. Let v_0 and v_1 be the adjacent vertices fixed by σ_0 and σ_1 respectively, afforded by Theorem D.3. The element $\gamma = \sigma_0^d \sigma_1^e$ maps v_1 to a vertex v' adjacent to v_0 , which is at distance 2 from v_0 in the 1-skeleton provided $d \neq 0$. Since the set $\{g \in \Gamma \mid \gamma(v_0) = v'\} = \{\sigma_0^d \sigma_1^e \mid e \in \mathbf{Z}/n\mathbf{Z}\}$, we deduce from the first assertion that if $d \neq 0$, there are exactly two values of e such that $\sigma_0^d \sigma_1^e$ is of finite order. Those values must be $e = 0$ or $e = \pi_1(d)$ since $\sigma_0^d \sigma_1^{\pi_1(d)} = \sigma_2^{-\pi_2(d)}$. \square

D.3 Maps between panel-regular lattices

A groundbreaking example of an exotic \tilde{A}_2 -building with a vertex-transitive lattice and non-Desarguesian residues has recently been constructed in [54]. That building is of order 9, and contains exotic sub-buildings of order 3. This feature suggests to study the existence of maps between \tilde{A}_2 -lattices in buildings of different orders. In the context of panel-regular lattices, we record the following.

Proposition D.5. *Let q, n, D and q_0, n_0, D_0 be two triples as in Theorem D.3. Let also $\tau_1, \tau_2 \in \text{Sym}(D_0)$ and $\pi_1, \pi_2 \in \text{Sym}(D)$ be permutations all fixing 0, and consider the groups*

$$\Gamma_0 = \langle s_0, s_1, s_2 \mid s_0^{n_0}, s_1^{n_0}, s_2^{n_0}, s_0^d s_1^{\tau_1(d)} s_2^{\tau_2(d)} \forall d \in D_0 \rangle$$

and

$$\Gamma = \langle \sigma_0, \sigma_1, \sigma_2 \mid \sigma_0^n, \sigma_1^n, \sigma_2^n, \sigma_0^d \sigma_1^{\pi_1(d)} \sigma_2^{\pi_2(d)} \forall d \in D \rangle,$$

and the associated buildings X_0 and X afforded by Theorem D.3.

Assume that the following conditions hold.

- n_0 divides n .
- $\{dn/n_0 \mid d \in D_0\} \subseteq D$.
- $\pi_i(dn/n_0) = \tau_i(d)n/n_0$ for all $d \in D_0$ and $i \in \{1, 2\}$.

Then the assignments $s_i \mapsto \sigma_i^{n/n_0}$ for $i = 0, 1, 2$ extend to a homomorphism $\Gamma_0 \rightarrow \Gamma$ with infinite image.

Proof. Set $\varphi(s_i) = \sigma_i^{n/n_0}$ for $i = 0, 1, 2$. Since n_0 divides n , we have $\varphi(s_i)^{n_0} = 1$ for all i . Moreover, given $d \in D_0$, we have

$$\begin{aligned} \varphi(s_0)^d \varphi(s_1)^{\tau_1(d)} \varphi(s_2)^{\tau_2(d)} &= \sigma_0^{dn/n_0} \sigma_1^{\tau_1(d)n/n_0} \sigma_2^{\tau_2(d)n/n_0} \\ &= \sigma_0^{dn/n_0} \sigma_1^{\pi_1(dn/n_0)} \sigma_2^{\pi_2(dn/n_0)} \\ &= 1 \end{aligned}$$

since dn/n_0 is an element of D by hypothesis. Thus the elements $\varphi(s_i)$ satisfy the defining relations of Γ_0 , so that φ indeed extends to a homomorphism $\varphi: \Gamma_0 \rightarrow \Gamma$. To prove that $\varphi(\Gamma_0)$ is infinite, we pick any non-zero $d \in D_0$. Then s_0^d and $s_0^d s_1^{\tau_1(d)}$ are both of finite order. By virtue of Lemma D.4, we deduce that for any $e \in \mathbf{Z}/n_0\mathbf{Z} \setminus \{0, \tau_1(d)\}$, the element $\varphi(s_0^d s_1^e) = \sigma_0^{dn/n_0} \sigma_1^{en/n_0}$ is of infinite order. \square

Remark D.6. As mentioned in the introduction, an unpublished work of Yehuda Shalom and Tim Steger ensures that every proper normal subgroup of a lattice in an \tilde{A}_2 -building is of finite index. In particular, the homomorphism $\Gamma_0 \rightarrow \Gamma$ constructed in Proposition D.5 must be injective. Using [33, Corollary 5.7], this implies that the embedding $\Gamma_0 \rightarrow \Gamma$ is accompanied by a Γ_0 -equivariant isometric embedding $X_0 \rightarrow X$. This noteworthy geometric feature will however not be needed in the subsequent discussion.

Invoking Theorem 1.1, we deduce the following exoticity criterion for X .

Proposition D.7. *In the setting of Proposition D.5, if X is Bruhat–Tits, then at least one of the following assertions holds:*

(I) X_0 is also Bruhat–Tits.

(II) Γ is a Galois lattice; moreover Γ_0 and Γ both have an infinite quotient which is $\{\text{residually } p\}\text{-by-cyclic}$, where p is the residue characteristic of the defining local field of X .

Proof. Assume that X is Bruhat–Tits, and let \mathbf{G} be the simple adjoint algebraic group over a local field k to which X is associated. By Proposition C.1(i), (v) and (vi), we have $\text{Aut}(X) = \text{Aut}(\mathbf{G}(k))$ and the group $\text{Aut}(\mathbf{G}(k))$ decomposes as an extension of $\text{Aut}(\mathbf{G})(k)$ by $\text{Aut}_{\mathbf{G}}(k)$, which is a subgroup of the full Galois group $\text{Aut}(k)$. Moreover, if k has characteristic 0 then $\text{Aut}(k)$ is finite, while if k has characteristic p , then $\text{Aut}(k)$ is an extension of a pro- p group by a finite cyclic group (see Theorem 1 and the remark following it in [18]). Therefore, composing the homomorphism $\varphi: \Gamma_0 \rightarrow \Gamma \leq \text{Aut}(X) = \text{Aut}(\mathbf{G}(k))$ with the quotient map $\text{Aut}(\mathbf{G}(k)) \rightarrow \text{Aut}_{\mathbf{G}}(k)$, we obtain a homomorphism whose image is $\{\text{residually } p\}\text{-by-cyclic}$. If that image is infinite, then Γ is a Galois lattice and Γ_0 and Γ have an infinite $\{\text{residually } p\}\text{-by-cyclic}$ quotient. Otherwise, using again Proposition C.1(i), the group Γ_0 has a finite index subgroup whose image under φ is contained in the group $\mathbf{G}(k) \leq \text{Aut}(X)$. Thus φ induces in particular a linear representation with infinite image of a finite index subgroup of Γ_0 . By Theorem 1.1, this implies that X_0 is Bruhat–Tits. \square

Remark D.8. As mentioned in Remark D.6, every proper quotient of Γ and Γ_0 is finite, so that in Case (II) of Proposition D.7, both Γ_0 and Γ must be $\{\text{residually } p\}\text{-by-cyclic}$. In particular, each of their finite index subgroups has a non-trivial abelianization. That condition will in particular exclude all the panel-regular lattices that are virtually perfect (see for example the lattice Γ_0 in Lemma D.9 below; many more perfect panel-regular lattices can be constructed using Theorem D.3).

On the other hand, a panel-regular lattice can be $\{\text{residually } p\}\text{-by-cyclic}$. A concrete example is provided by the group

$$\Gamma_2 = \langle s_0, s_1, s_2 \mid s_0^7, s_1^7, s_2^7, s_0 s_1 s_2, s_0^3 s_1^3 s_2^3 \rangle$$

associated with the difference set $\{0, 1, 3\}$ in $\mathbf{Z}/7\mathbf{Z}$. As pointed out by J. Essert [33, Remark following Th. 5.8], that group is an index 3 subgroup of an arithmetic lattice U in the group $G = \text{SL}_3(\mathbf{F}_2((t)))$ considered by Köhler–Meixner–Wester [45]. The lattice U is in fact contained in $\text{SL}_3(\mathbf{F}_2[t, (t^2 + t + 1)^{-1}])$ (see [45, Theorem 1]), and it is pointed out in [45, §3] that the image of Γ_2 under the congruence quotient induced by the evaluation $t \mapsto 0$ is cyclic of order 7. By Platonov’s theorem [52], the kernel of that congruence quotient is residually 2. Thus Γ_2 is $\{\text{residually } 2\}\text{-by-}\{\text{cyclic of order } 7\}$.

D.4 Examples in characteristic 2

We shall now specify a choice of Γ_0 which is suitable for an application of Proposition D.7.

Lemma D.9. *Consider the set $D_0 = \{0, 1, 3\}$, which is a difference set in $\mathbf{Z}/7\mathbf{Z}$, and the panel-regular lattice*

$$\Gamma_0 = \langle s_0, s_1, s_2 \mid s_0^7, s_1^7, s_2^7, s_0 s_1 s_2^3, s_0^3 s_1^3 s_2 \rangle$$

and the associated \tilde{A}_2 -building X_0 of order 2 afforded by Theorem D.3. Then:

- (i) X_0 is not Bruhat–Tits.
- (ii) The abelianization $\Gamma_0/[\Gamma_0, \Gamma_0]$ has order 7, and the derived group $[\Gamma_0, \Gamma_0]$ is perfect.
- (iii) The assignments $s_0 \mapsto s_0^2$, $s_1 \mapsto s_1^2$, $s_2 \mapsto s_0 s_1$ extend to an automorphism $\alpha \in \text{Aut}(\Gamma_0)$ of order 3. The extension $\Gamma_0 \rtimes \langle \alpha \rangle$ acts freely and transitively on the set of chambers of X_0 (and coincides with the group G_4 from [58, Th. 2.5]).

Proof. By [58, Theorem 2.5], there are exactly 4 possible groups acting freely and transitively on the chambers of an \tilde{A}_2 -building, denoted there by G_1, G_2, G_3 and G_4 . (Those groups and their structure are also discussed in [68], [69], [70], [43, §C.3] and [46].) Using the explicit presentations from [43, Theorem C.3.6], one sees that G_4 is isomorphic to $\Gamma_0 \rtimes \langle \alpha \rangle$. This proves (iii). Assertion (i) follows from [70, Cor. 1], and assertion (ii) from [46, Prop. 3]. \square

Corollary D.10. *Given q a power of 2 which is not a power of 8, there exists a panel-regular lattice Γ whose building X has order q and is not Bruhat–Tits.*

Proof. Let $n = q^2 + q + 1$. By Lemma D.2, there is a difference set $D \subset \mathbf{Z}/n\mathbf{Z}$ containing $\{0, n/7, 3n/7\}$.

We pick permutations $\pi_1, \pi_2 \in \text{Sym}(D)$ fixing 0 such that $\pi_1(n/7) = n/7$, $\pi_1(3n/7) = 3n/7$, $\pi_2(n/7) = 3n/7$ and $\pi_2(3n/7) = n/7$. Let then Γ be the panel-regular lattice associated with D , π_1 and π_2 via Theorem D.3. Its building X has order q . Moreover, Proposition D.5 provides a homomorphism $\Gamma_0 \rightarrow \Gamma$, where Γ_0 is the panel-regular lattice from Lemma D.9. Since Γ_0 has a perfect group of index 7, for any prime p , every $\{\text{residually } p\}$ -by-cyclic quotient of Γ_0 has order at most 7. Therefore, Proposition D.7 ensures that X is not Bruhat–Tits. \square

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